Fundamentals of Engineering Review for Dynamics

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&

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Review Notes available in PDF format @
http://www.eng.lsu.edu/students/current/feinfo.html

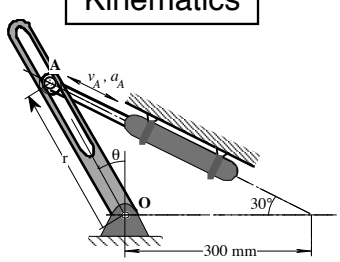
Last Revised: 02/14

Louisiana State University

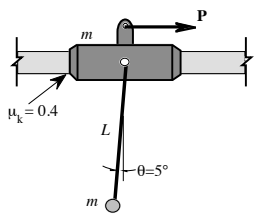
Dynamics Problem Decomposition

Dynamics

Kinematics | Kinetics



Geometric descriptions of motion & constraints

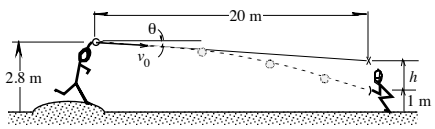


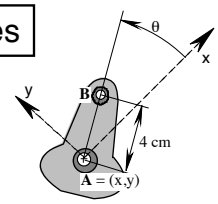
Loading relationships which dictate CHANGES in motion

Dynamic Studies

Dynamics:
(Kinematics & Kinetics)

Particles | Rigid Bodies



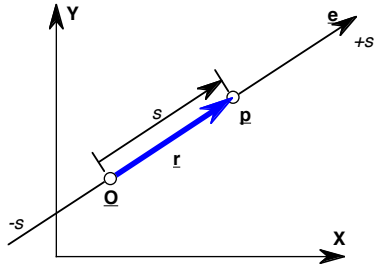


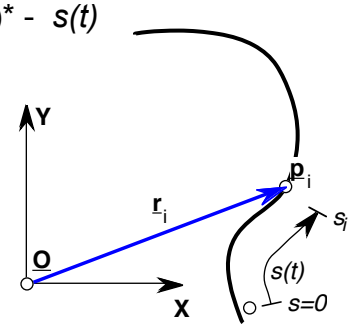
- **M** - mass
- (x,y) - position in 2D
- 2 Degrees of Freedom (DOF) in the plane

- **M** & ***I*** - both mass & rotational inertia
- (x,y,θ) - position & orientation in 2D (3DOF)

Getting Started => Particle Kinematics

- Rectilinear Motion
  - Movement along a straight line in 1-2 or 3D
    - 1 Degree of Freedom (DOF)* - *s(t)*





- Curvilinear Motion
  - Movement of particle along an arbitrary path through space

Rectilinear Motion Overview (Calculus/Physics Review!):

DIFFERENTIATE

INTEGRATE

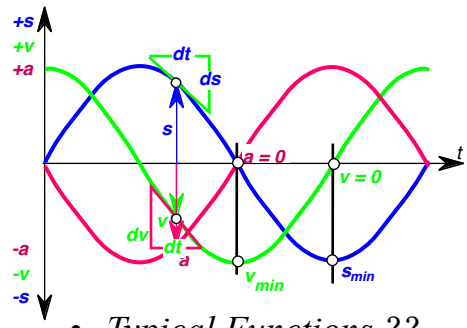
- Position - $s(t)$
- Speed - $v(t)$

(1) $v = \dfrac{ds}{dt} = \dot{s}$

- Acceleration - $a(t)$

(2) $a = \dfrac{dv}{dt} = \dot{v} = \dfrac{d^2s}{dt^2} = \ddot{s}$

- ***Basic Calculus!***
  - Max's & Min's ?
  - Undo Differentiation?



- *Typical Functions ??*
  - Polynomials
  - Trigonometric
  - Logarithms
  - Exponentials

Rectilinear Motion Summary:

DIFFERENTIATE

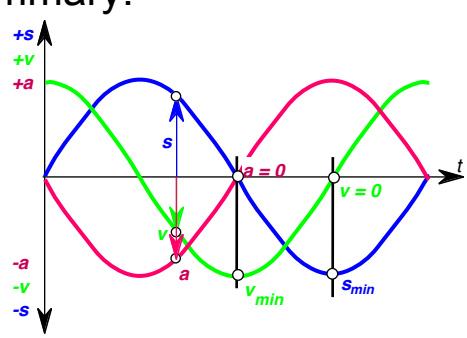
INTEGRATE

- Position - $s(t)$
- Speed - $v(t)$

(1) $v = \dfrac{ds}{dt} = \dot{s}$

- Acceleration - $a(t)$

(2) $a = \dfrac{dv}{dt} = \dot{v} = \dfrac{d^2s}{dt^2} = \ddot{s}$



- Alternate form ?

(2*) $v\ dv = a\ ds$

---

a(t) => Solid Rocket Propulsion *a(v)* => aerodynamic drag

a(s) => Gravitational fields, springs, conservative forces etc.

Given: $s = 2t^2 - 8t + 3$

Find: **Displacement from t = 1 to t = 3**
Distance traveled from t = 1 to t = 3

$s = 2t^2 - 8t + 3$ $\quad s(1) = -3 \;\&\; s(3) = -3 \quad s(2) = -5$

$v = \dot{s} = 4t - 8 \quad v(2) = 0$

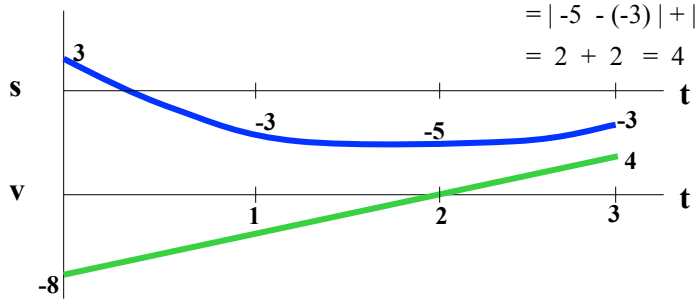
$a = \ddot{s} = 4 \quad a(2) = 4$

Since, $s(3) - s(1) = 0 \;\rightarrow\;$ displacement = 0

Reversal @ $v|_{t=2} = 0$ distance traveled $= |s(2) - s(1)| + |s(3) - s(2)|$

$= |-5 - (-3)| + |(-3) - (-5)|$

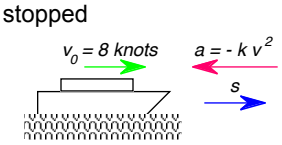
$= 2 + 2 = 4$



Rectilinear Kinematics: Accel. a function of velocity – a(v)

Given:

- A freighter moving at 8 knots when engines are stopped
- Deceleration a= -kv^2
- Speed reduces to 4 knots after ten minutes



Find:

(A) Speed of the ship as a function of time v(t)

(B) How far does the ship travel in the 10 minutes it takes to reduce the speed by 1/2 ?

Solution:

(A) With a, v & t parameters given/requested, use *a=dv/dt* form

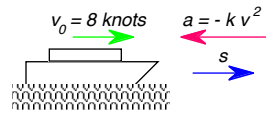
$$t_f = t(v) = \int_{v_i}^{v_f} \frac{dv}{a(v)} - t_i = \int_8^{v_f} \frac{dv}{-kv^2} + 0$$

$$\Rightarrow t_f = \frac{1}{kv}\bigg|_8^{v_f} = \frac{1}{k}\left(\frac{1}{v_f} - \frac{1}{8}\right) \Rightarrow v_f = v(t_f) = \frac{8}{8kt_f + 1} \quad (knots)$$

Rectilinear Kinematics: Accel. a function of velocity – a(v)

- Substituting BC's helps resolve the unknown constant *k*

$$t = \frac{10\ (min)}{60\ (min/hr)} = \frac{1}{6}\ hr\ ,\ v = 4\ \ knots$$



$$\Rightarrow v(1/6) = \frac{8}{8k(1/6)+1} = 4\ (knots)\ \ \Rightarrow \mathrm{k} = \frac{3}{4}\left(\frac{1}{\mathrm{nm}}\right)$$

and the resulting expression for *speed* of the ship as a function of time v(t) is as follows

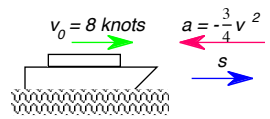
$$\underline{\underline{v_f = v(t) = \frac{8}{6t+1}}}\quad (knots)$$

- From here, there are two alternatives for resolving the second question

Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 1: Now, knowing the velocity as a function of time

$$v(t) = \frac{8}{6t+1} = \frac{ds}{dt}$$



the boat's position can be found by integration

$$\int_0^{s_f} ds = \int_0^{t_f} \frac{8}{6t+1}\,dt$$

$$s_f - 0 = \frac{4}{3}\ \ln(6t+1)\Big|_0^{t_f} = \frac{4}{3}\ \left(\ln(6t+1) - \ln(1)\right)$$

and the resulting expression for *position* of the ship as a function of time s(t)

$$s_f = s(t) = \frac{4}{3}\ \ln(6t+1)$$

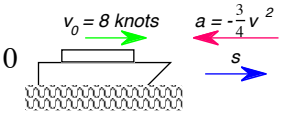
can now be used to find the particular displacement/distance at t=1/6 hr !

$$s(1/6) = \frac{4}{3}\ \ln(6(1/6)+1) = \underline{\underline{\frac{4}{3}\ \ln(2)}}\quad (nautical\ \ miles)$$

Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 2: With a, v & s parameters given/requested, use *ads=vdv* form

$$s_f = s(v) = \int_{v_i}^{v_f} \frac{vdv}{a(v)} + s_i \Rightarrow s(4) = \int_8^4 \frac{vdv}{\left(-3/4v^2\right)} + 0$$



and the boat's displacement (position?) can again be found by integration

$$s(4) = \frac{-4}{3}\int_8^4 \frac{dv}{v} = \frac{-4}{3} ln v \Big|_8^4 = \frac{-4}{3}\left(ln4 - ln8\right) = \frac{4}{3} ln\frac{8}{4}$$

and as was seen before

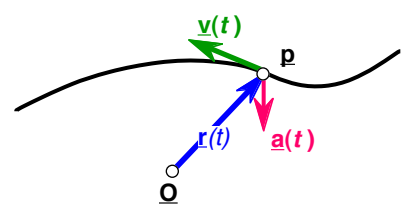
$$s(t = 1/6) \Rightarrow s(v = 4) = \underline{\underline{\frac{4}{3}\ \ln(2)}} \quad (nautical\ \ miles)$$

Q.E.D.

Curvilinear Kinematics Summary:

- Position

$$\underline{\mathbf{r}}(t) = x(t)\underline{\mathbf{i}} + y(t)\underline{\mathbf{j}} + z(t)\underline{\mathbf{k}}$$
$$= r(t)\underline{\mathbf{e}}_r + z\underline{\mathbf{e}}_z$$



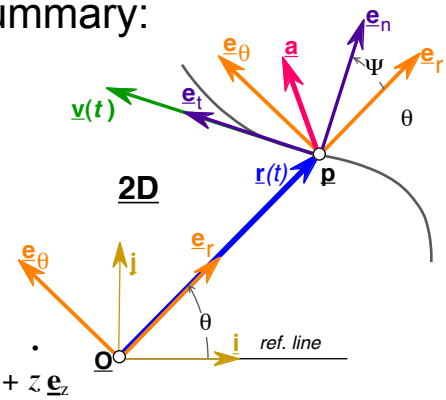
- Velocity: $\underline{\mathbf{v}} = d\,\underline{\mathbf{r}} / dt$

$$\underline{\mathbf{v}}(t) = \dot{\underline{\mathbf{r}}}(t) = \dot{x}\underline{\mathbf{i}} + \dot{y}\underline{\mathbf{j}} + \dot{z}\underline{\mathbf{k}}$$
$$= v\underline{\mathbf{e}}_t = \dot{s}\underline{\mathbf{e}}_t = \dot{r}\underline{\mathbf{e}}_r + r\dot{\theta}\,\underline{\mathbf{e}}_\theta + \dot{z}\,\underline{\mathbf{e}}_z$$

- Acceleration: $\underline{\mathbf{a}} = d\,\underline{\mathbf{v}} / dt$

$$\underline{\mathbf{a}}(t) = \dot{\underline{\mathbf{v}}}(t) = \ddot{\underline{\mathbf{r}}}(t) = \ddot{x}\underline{\mathbf{i}} + \ddot{y}\underline{\mathbf{j}} + \ddot{z}\underline{\mathbf{k}} = \dot{v}\underline{\mathbf{e}}_t + \rho\dot{\theta}^2\,\underline{\mathbf{e}}_n = \dot{v}\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\underline{\mathbf{e}}_n$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{\mathbf{e}}_\theta + \ddot{z}\,\underline{\mathbf{e}}_z$$

Curvilinear Kinematics Summary:



- Position

$$\underline{\mathbf{r}}(t) = x(t)\underline{\mathbf{i}} + y(t)\underline{\mathbf{j}} + z(t)\underline{\mathbf{k}}$$
$$= r(t)\underline{\mathbf{e}}_r + z\,\underline{\mathbf{e}}_z$$

- Velocity: $\underline{\mathbf{v}} = d\,\underline{\mathbf{r}} / dt$

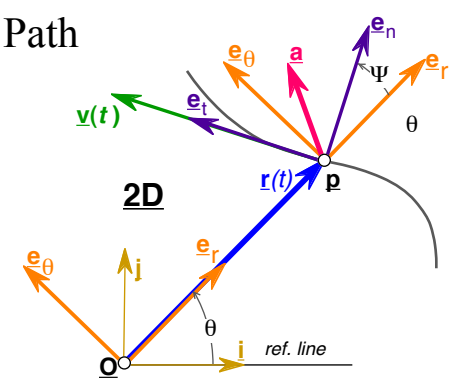
$$\underline{\mathbf{v}}(t) = \dot{\underline{\mathbf{r}}}(t) = \dot{x}\underline{\mathbf{i}} + \dot{y}\underline{\mathbf{j}} + \dot{z}\underline{\mathbf{k}}$$
$$= v\underline{\mathbf{e}}_t = \dot{s}\underline{\mathbf{e}}_t = \dot{r}\underline{\mathbf{e}}_r + r\dot{\theta}\,\underline{\mathbf{e}}_\theta + \dot{z}\,\underline{\mathbf{e}}_z$$

- Acceleration: $\underline{\mathbf{a}} = d\,\underline{\mathbf{v}} / dt$

$$\underline{\mathbf{a}}(t) = \dot{\underline{\mathbf{v}}}(t) = \ddot{\underline{\mathbf{r}}}(t) = \ddot{x}\underline{\mathbf{i}} + \ddot{y}\underline{\mathbf{j}} + \ddot{z}\underline{\mathbf{k}} = \dot{v}\underline{\mathbf{e}}_t + \rho\dot{\theta}^2\,\underline{\mathbf{e}}_n = \dot{v}\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\underline{\mathbf{e}}_n$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{\mathbf{e}}_\theta + \ddot{z}\,\underline{\mathbf{e}}_z$$

2D Curvilinear Motion: Coordinates & Conversions

- Cartesian <-> Polar <-> Path



$$\underline{\mathbf{r}}(t) = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} = r\underline{\mathbf{e}}_r$$
$$\underline{\mathbf{e}}_r = \cos\theta\underline{\mathbf{i}} + \sin\theta\underline{\mathbf{j}} = \frac{x}{r}\underline{\mathbf{i}} + \frac{y}{r}\underline{\mathbf{j}}$$
$$\underline{\mathbf{e}}_\theta = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_r = \cos\theta\underline{\mathbf{j}} - \sin\theta\underline{\mathbf{i}}$$
$$\underline{\mathbf{i}} = \cos\theta\underline{\mathbf{e}}_r - \sin\theta\underline{\mathbf{e}}_\theta$$
$$\underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{i}} = \cos\theta\underline{\mathbf{e}}_\theta + \sin\theta\underline{\mathbf{e}}_r$$

$$\underline{\mathbf{v}}(t) = \dot{\underline{\mathbf{r}}}(t) = \dot{x}\underline{\mathbf{i}} + \dot{y}\underline{\mathbf{j}} = v\underline{\mathbf{e}}_t = \dot{s}\underline{\mathbf{e}}_t$$
$$\underline{\mathbf{e}}_t = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|} = \frac{\dot{x}}{v}\underline{\mathbf{i}} + \frac{\dot{y}}{v}\underline{\mathbf{j}}$$
$$\underline{\mathbf{e}}_n = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_t = \frac{\dot{x}}{v}\underline{\mathbf{j}} - \frac{\dot{y}}{v}\underline{\mathbf{i}}$$

$$\underline{\mathbf{i}} = \cos\Psi\underline{\mathbf{e}}_t - \sin\Psi\underline{\mathbf{e}}_n$$
$$\underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{i}} = \cos\Psi\underline{\mathbf{e}}_n + \sin\Psi\underline{\mathbf{e}}_t$$

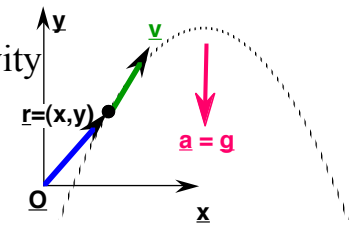
Curvilinear Motion: Cartesian Coordinates

- Projectile Motion
  - Scale w.r.t. earth such that gravity $\underline{\mathbf{g}}$ is ~<u>constant</u>

    $|g| = 32.2$ ft/s^2 = 9.81 m/s^2
  - Neglect any air resistance
  - Motion is PARABOLIC thus PLANAR!
  - Typically align
    - *y*-axis along gravity vector
    - *x*-axis horizontal in direction of motion

$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g]$$

    - *z component drops out!*



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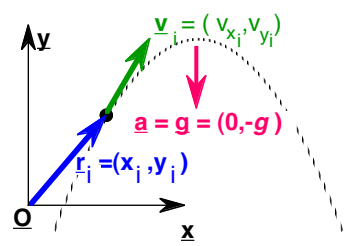
Curvilinear Motion: Projectile Motion

- Integrate rectilinear relations
  - Two (2) scalar relations
  - One VECTOR relationship

$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g] = \underline{\mathbf{a}}_c$$

$$\Rightarrow \int_{\underline{\mathbf{v}}_i}^{\underline{\mathbf{v}}_f} d\underline{\mathbf{v}} = \int_{t_i}^{t_f} \underline{\mathbf{a}}_c\, dt$$



$$\left(\underline{\mathbf{v}}_f - \underline{\mathbf{v}}_i\right) = \underline{\mathbf{a}}_c\left(t_f - t_i\right) \qquad \Rightarrow \int_{\underline{\mathbf{r}}_i}^{\underline{\mathbf{r}}_f} d\underline{\mathbf{r}} = \int_{t_i}^{t_f} \underline{\mathbf{v}}(t)\,dt$$

$$\Rightarrow \underline{\underline{\underline{\mathbf{v}}}_f(t) = \underline{\mathbf{a}}_c\left(t_f - t_i\right) + \underline{\mathbf{v}}_i} \qquad \Rightarrow \underline{\underline{\underline{\mathbf{r}}}_f = \frac{\underline{\mathbf{a}}_c}{2}\left(t_f - t_i\right)^2 + \underline{\mathbf{v}}_i\left(t_f - t_i\right) + \underline{\mathbf{r}}_i}$$

$$v_{x_f} = 0\left(t_f - t_i\right) + v_{x_i} = v_{x_i} \qquad x_f = v_{xi}\left(t_f - t_i\right) + x_i$$

$$v_{y_f} = -g\left(t_f - t_i\right) + v_{y_i} \qquad y_f = \frac{-g}{2}\left(t_f - t_i\right)^2 + v_{y_i}\left(t_f - t_i\right) + y_i$$

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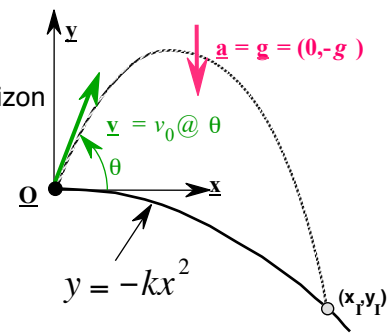
Curvilinear Kinematics: Projectile Motion example ref: Hibbler 12-104

Given:

- Figure shown w/ ground $y = -kx^2$
- t_0=0, $(x_0,y_0)=\underline{0}$, $\underline{\mathbf{v}}_0=v_0$ @ θ above horizon



Find: In terms of $v0$, θ & k

- (A) The location at impact (x_I,y_I)
- (B) Velocity & Speed @ impact, $\underline{\mathbf{v}}_I$, v_I
- (C) Elapsed time @ impact, t_I

Solution:

- 2D projectile motion
- Get expressions for $v_x(t)$,$v_y(t)$ then $x(t)$, $y(t)$
- Substitute into ground constraint expression
  - Solve for time of impact
- With t_I known, substitute & solve for (x_I,y_I)

$$\Rightarrow \underline{\mathbf{v}}_f(t) = \underline{\mathbf{a}}_c\left(t_f - t_i\right) + \underline{\mathbf{v}}_i \qquad \Rightarrow \underline{\mathbf{r}}_f = \frac{\underline{\mathbf{a}}_c}{2}\left(t_f - t_i\right)^2 + \underline{\mathbf{v}}_i\left(t_f - t_i\right) + \underline{\mathbf{r}}_i$$

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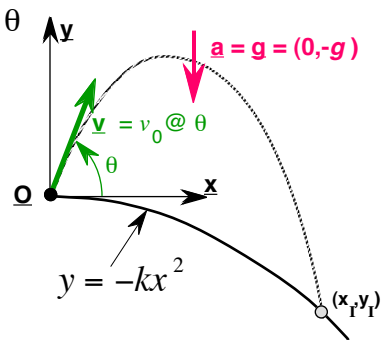
Curvilinear Kinematics: Projectile Motion

- IC's => t_0=0, $(x_0,y_0)=\underline{0}$, $\underline{\mathbf{v}}_0=v_0$ @ θ

$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g]$$



$$\Rightarrow (B) \quad \underline{\mathbf{v}}_I = \underline{\mathbf{a}}\left(t_I - 0\right) + \underline{\mathbf{v}}_0 = \left[v_{x_I}, v_{y_I}\right]$$

$$v_{x_I} = v_0 \cos\theta$$

$$v_{y_I} = -gt_I + v_0 \sin\theta$$

- Speed

$$\dot{s} = v = \sqrt{v_{x_I}^2 + v_{y_I}^2}$$

$$= \sqrt{\left(v_0 \cos\theta\right)^2 + \left(-gt_I + v_0 \sin\theta\right)^2}$$

$$= \sqrt{v_0^2 - 2gv_0 \sin\theta t_I + \left(gt_I\right)^2}$$

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Curvilinear Kinematics: Projectile Motion

$$\Rightarrow (B) \quad \underline{\mathbf{v}}_I = \underline{\mathbf{a}}(t_I - 0) + \underline{\mathbf{v}}_0 = \left[v_{x_I}, v_{y_I} \right]$$

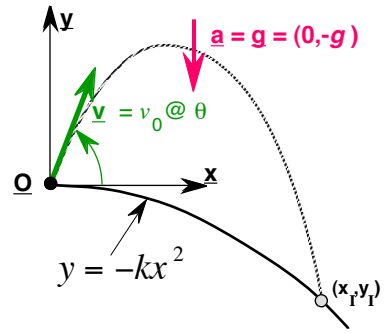
$$\Rightarrow (A) \quad \underline{\mathbf{r}}_I = \frac{\underline{\mathbf{a}}}{2}(t_I - 0)^2 + \underline{\mathbf{v}}_0(t_I - 0) + \underline{\mathbf{0}}$$

$$\begin{cases} x_I = v_0 \cos\theta \; t_I \\ y_I = \frac{-g}{2} t_I^2 + v_0 \sin\theta \; t_I \end{cases}$$

$$y = -kx^2$$

$$\frac{-g}{2} t_I^2 + v_0 \sin\theta \; t_I = -k\left(v_0 \cos\theta \; t_I\right)^2$$

$$\Rightarrow (C) \quad t_I = \frac{2 v_0 \sin\theta}{g - 2k\left(v_0 \cos\theta\right)^2} \quad , \quad t_I = 0$$



– Substitute value for t_I into position, velocity & speed relations for solution

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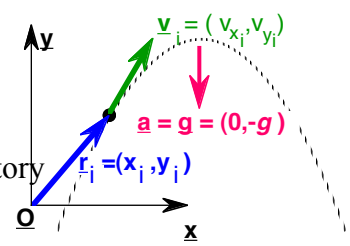
Curvilinear Motion: Projectile Motion

– Other typical P.M. queries
- Max Height
- Max Range
- Time @ some place along trajectory
- Later w/ Path & Polar Coord
  - Velocity (speed,direction/tangent)
  - Curvature, rate of speed change ….



$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g]$$

$$\Rightarrow \underline{\mathbf{v}}_f = \underline{\mathbf{a}}(t_f - t_i) + \underline{\mathbf{v}}_i \qquad \Rightarrow \underline{\mathbf{r}}_f = \frac{\underline{\mathbf{a}}}{2}(t_f - t_i)^2 + \underline{\mathbf{v}}_i(t_f - t_i) + \underline{\mathbf{r}}_i$$

– Reconsider problems w/ different axes placement

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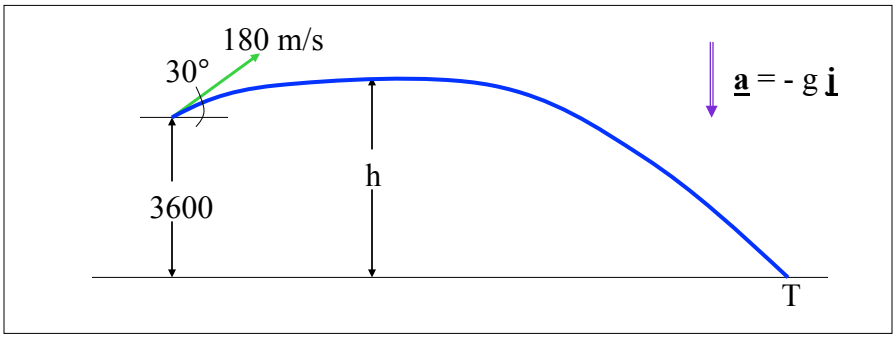
Given: **launch at 3600 m altitude v_o = 180 m/s angle 30°**



$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -9.81$

$\dot{x} = 180\ (\cos 30) = 156$ $\qquad$ $\dot{y} = 180\ (\sin 30) - 9.81\ t = 90 - 9.81\ t$

$x = 156\ t$ $\qquad$ $y = 90\ t - 4.905\ t^2 + 3600$

for h set $\quad \dot{y} = 0 \implies t = 9.17 \implies h = y = 4013$ m

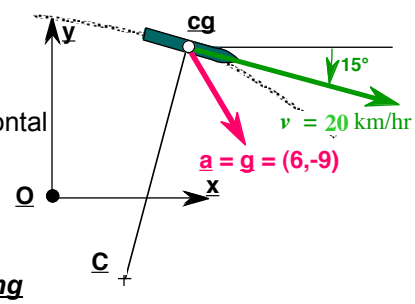
for t_T set $\quad y = 0 \implies t = 31.18$

21

Path Coord. Example ref: Meriam&Kraige 2-8

Given:

- A rocket at high altitude with
- $\underline{a}_0 = 6\underline{i} - 9\underline{j}$ (m/s^2)
- $\underline{v}_0 = 20$ (km/hr) @ 15° below horizontal



Find: At instant given

(A) The ***normal*** & ***tangential*** *accelerations*

(B) Rate at which ***speed*** is ***increasing***

(C) ***Radius of curvature*** of the path

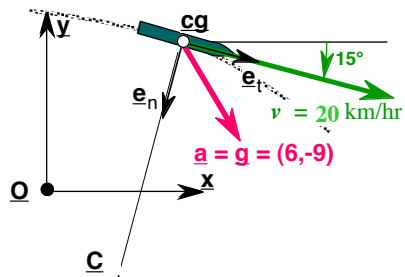
(D) Angular ***rotation rate*** of the radial from CG to center of curvature

Solution:

- "High altitude" means negligible air resistance
- Interested only at this instant (NO Integration required)
- **V** given is TANGENT TO THE PATH
  - Use this to relate path to cartesian coordinates

Path Coord. Example ref: Meriam&Kraige 2-8

Solution (cont'd):



$$\underline{e}_t = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{v}}{v} = \cos 15°\underline{i} - \sin 15°\underline{j}$$

$$\underline{e}_n = \pm\left(\underline{k} \times \underline{e}_t\right) \qquad \textit{(2D shortcut!)}$$

$$= -\cos 15°\underline{j} - \sin 15°\underline{i}$$

(A) a_n & a_t=?

$$|\underline{a}_t| = \underline{a} \bullet \underline{e}_t = \left(6\underline{i} - 9\underline{j}\right) \bullet \left(\cos 15°\underline{i} - \sin 15°\underline{j}\right) = \underline{\underline{8.12\ (m/s^2) = a_t}}$$

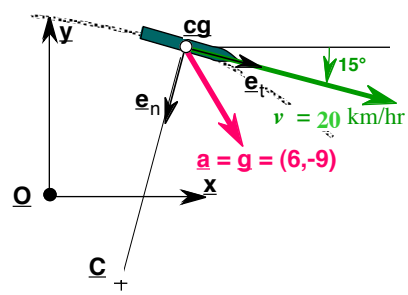
$$|\underline{a}_n| = \underline{a} \bullet \underline{e}_n = \left(6\underline{i} - 9\underline{j}\right) \bullet \left(-\cos 15°\underline{j} - \sin 15°\underline{i}\right) = \underline{\underline{7.14\ (m/s^2) = a_n}}$$

(B) $\dot{v} = ??$ $\quad \dot{v} = |\underline{a}_t| = \underline{\underline{8.12\ (m/s^2)}}$

(C) ρ=? $a_n = \dfrac{v^2}{\rho} \Rightarrow \rho = \dfrac{v^2}{a_n} = \dfrac{(20\ km/hr)^2}{7.14(m/s^2)}\left(\dfrac{1hr}{3600s} * \dfrac{10^3 m}{km}\right)^2 = \underline{\underline{4.32(10^6)m}}$

Path Coord. Example ref: Meriam&Kraige 2-8

Solution (cont'd):



(D) $\dot{\theta} = ??$

– Look either at a_n or velocity

$$a_n = \rho\dot{\theta}^2$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{a_n}{\rho}} = \sqrt{\frac{7.14\ (m/s^2)}{4.32 * 10^6 (m)}} = \underline{\underline{12.9 * 10^{-4}\ \frac{1}{s}}}$$

$$v = \rho\dot{\theta}$$

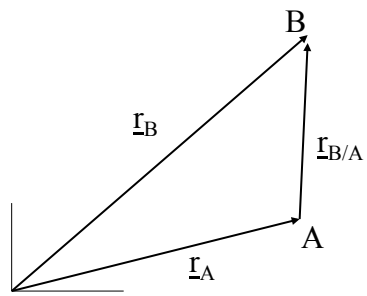
$$\Rightarrow \dot{\theta} = \frac{v}{\rho} = \frac{20\ km/hr}{4.32 * 10^6 (m)}\left(\frac{1hr}{3600s} * \frac{10^3 m}{km}\right) = \underline{\underline{12.9 * 10^{-4} m}}$$

RELATIVE MOTION

$\underline{\mathbf{r}}_B = \underline{\mathbf{r}}_A + \underline{\mathbf{r}}_{B/A}$

$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$

$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\mathbf{a}}_{B/A}$

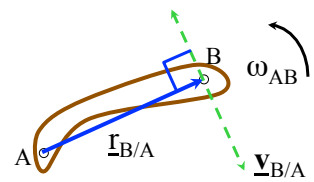


Special Case: Rigid Bodies

When A & B are two points on the same rigid body then:

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \omega_{AB}\ \underline{\mathbf{k}} \times AB\underline{\mathbf{u}}_{B/A} \quad (2D)$$

- i.e. the relative motion is circular
- $\underline{\mathbf{v}}_{B/A}$ is perpendicular ($\perp$) to $\underline{\mathbf{r}}_{B/A}$ &
- $|\ \underline{\mathbf{v}}_{B/A}\ | = |\ \omega_{AB}\ AB\ |$



Relative Motion: ref ~DWY FE Review

Given:

- A river flows south at 5 m/s
- A boat can travel at 10 m/s relative to the water.

Find:

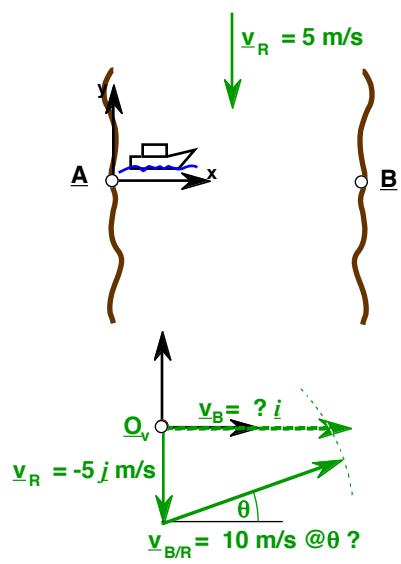
- In what *direction* should the boat head from A, in order to reach point B, directly across the river?

Solution:

- Set up Cartesian Axes @A
- Write Relative Velocity expression

$$\underset{\text{DM}}{\underline{\mathbf{V}}_B} = \underset{\text{DM}}{\underline{\mathbf{V}}_R} + \underset{\text{DM}}{\underline{\mathbf{V}}_{B/R}}$$

- Draw *Velocity Polygon* diagram
  - $\underline{\mathbf{V}}_R$ *magnitude* & *direction* known
  - $\underline{\mathbf{V}}_B$ *direction* known, not *magnitude*
  - $\underline{\mathbf{V}}_{B/R}$ *magnitude* known, not *direction*



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Relative Motion: ref ~DWY/WNW FE Review

Solution (cont' d):

$\underline{\mathbf{V}}_B = \underline{\mathbf{V}}_R + \underline{\mathbf{V}}_{B/R}$

$v_B\,\underline{\mathbf{i}} = v_R\,\underline{\mathbf{j}} + v_{B/R}\,(\cos\theta\,\underline{\mathbf{i}} + \sin\theta\,\underline{\mathbf{j}})$

$v_B\,\underline{\mathbf{i}} = -5\,\underline{\mathbf{j}} + 10\,(\cos\theta\,\underline{\mathbf{i}} + \sin\theta\,\underline{\mathbf{j}})$ (m/s)

- Equating components:

  $\underline{\mathbf{i}}$ => $v_B = 10\cos\theta$ (m/s)

  $\underline{\mathbf{j}}$ => $0 = -5 + 10\sin\theta$ (m/s)

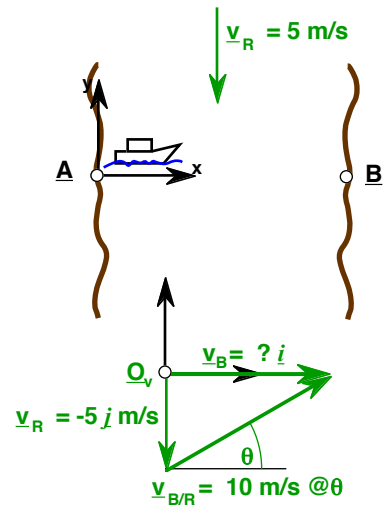
- 2 equations <-> two unknowns

  $\sin\theta = 5/10 = 1/2$ (m/s)

  => $\underline{\theta = \sin^{-1}(1/2) = 30°}$

- Lagniappe

  $\underline{v_B = 10\cos 30°}$ (m/s)



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Relative Motion: ref ~DWY/WNW FE Review

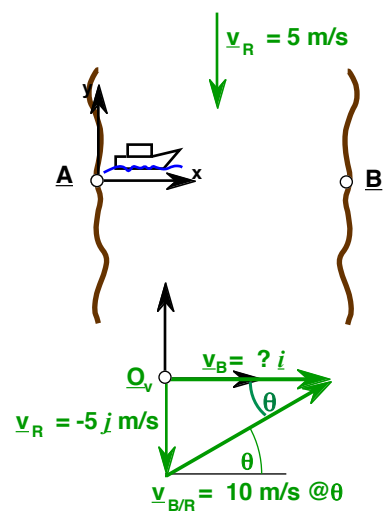
Alternate Solution (cont' d):

- With velocity polygon drawn, use the Law of Sines

$$\frac{\sin\theta}{|v_R|} = \frac{\sin 90°}{|v_{B/R}|}$$

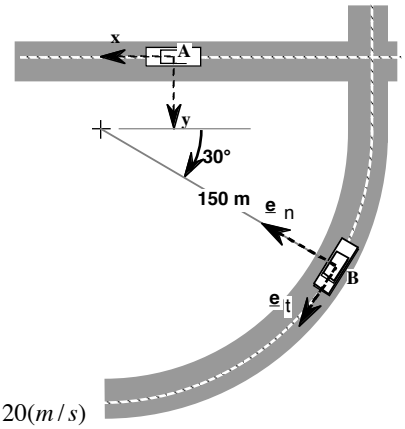
$$\theta = \sin^{-1}\frac{|v_R|\sin 90°}{|v_{B/R}|}$$

$$= \sin^{-1}\left(\frac{5}{10}\right) = 30°$$



28

Relative Motion: ref ~Meriam & Kraige 2/13



Given:

- Two cars **A** & **B** at the instant shown

  $\underline{\mathbf{v}}_A$= 72 $\underline{\mathbf{i}}$ km/hr $\qquad$ $\underline{\mathbf{a}}_A$= 1.2 $\underline{\mathbf{i}}$ m/s^2

  $\underline{\mathbf{v}}_B$= 54 $\underline{\mathbf{e}}_t$ km/hr, *constant speed*

Find:

(A) $\underline{\mathbf{v}}_{B/A}$ =? $\quad$ (B) $\underline{\mathbf{a}}_{B/A}$ =?

Solution:

- Convert to consistent units

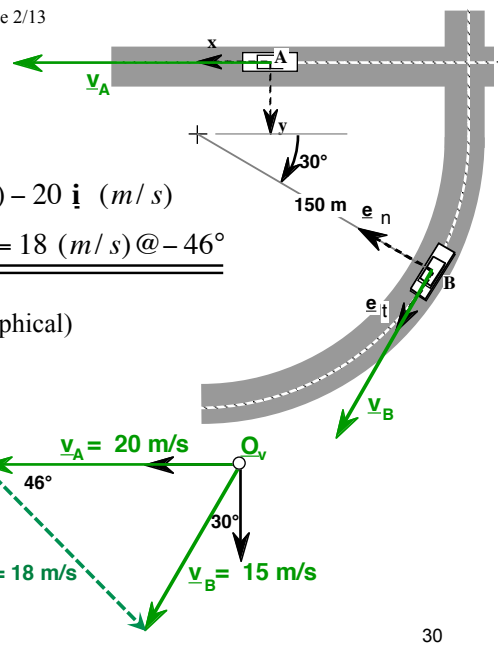
$$(km/hr)*\frac{1}{3.6} = (m/s) \Rightarrow \quad v_A = 72(km/hr) = 20(m/s)$$
$$v_B = 54(km/hr) = 15(m/s)$$

- Motion RELATIVE TO **A** of interest
- Two coordinate axes are used
  - Simplifies $\underline{\mathbf{v}}$ & $\underline{\mathbf{a}}$ definitions
  - Illustrates "coordinate conversion" for expressing answers "in terms of" a unified set.

$$\underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{i}} - \sin 30°\underline{\mathbf{j}}$$
$$\underline{\mathbf{e}}_t = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{j}} + \sin 30°\underline{\mathbf{i}}$$

29

Relative Motion: ref ~Meriam & Kraige 2/13



(A) Relative Velocity

$$\underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A$$
$$= 15\underline{\mathbf{e}}_t - 20\ \underline{\mathbf{i}}\ \ (m/s)$$
$$= 15(\sin 30°\underline{\mathbf{i}} + \cos 30°\underline{\mathbf{j}}) - 20\ \underline{\mathbf{i}}\ \ (m/s)$$
$$\underline{\underline{\mathbf{v}}_{B/A} = -12.5\underline{\mathbf{i}} + 13.0\underline{\mathbf{j}}\ (m/s) = 18\ (m/s) @ -46°}$$

- Velocity Polygon Approach (Graphical)

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$$
$$\Rightarrow \underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A$$

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Relative Motion: ref ~Meriam & Kraige 2/13

(B) Relative Acceleration

$$\underline{\mathbf{a}}_A = 1.2\,\underline{\mathbf{i}}\ \ (m/s^2)$$

$$\underline{\mathbf{a}}_B = \dot{v}\,\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\,\underline{\mathbf{e}}_n\ \ (m/s^2) = \frac{(15\ \ m/s)^2}{150\ \ m}\underline{\mathbf{e}}_n$$

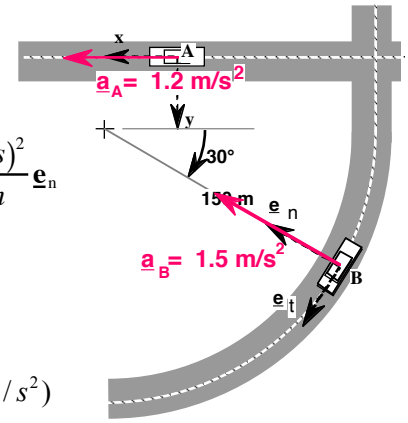
$$= 1.5\underline{\mathbf{e}}_n\ \ (m/s^2)$$

$$\underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A$$

$$= 1.5\,\underline{\mathbf{e}}_n - 1.2\underline{\mathbf{i}}\ \ (m/s^2)$$

$$= 1.5(\cos 30°\underline{\mathbf{i}} - \sin 30°\underline{\mathbf{j}}) - 1.2\underline{\mathbf{i}}\ \ (m/s^2)$$

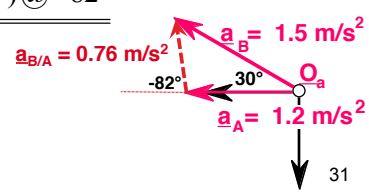
$$\underline{\mathbf{a}}_{B/A} = 0.1\,\underline{\mathbf{i}} - 0.75\,\underline{\mathbf{j}}\,(m/s^2) = 0.76(m/s^2)@-82°$$



- Acceleration Polygon (Graphical)

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\mathbf{a}}_{B/A}$$

$$\Rightarrow \underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A$$



31

Given: **A balloon at an altitude of 60 m is rising at steady rate of 4.5 m/s. A car passes below at constant speed of 72 kph.**

Find: **Relative rate of separation 1 second later:**

$\underline{\mathbf{a}}_C = 0 \qquad \underline{\mathbf{a}}_B = 0 \quad (m/s^2)$

$\underline{\mathbf{v}}_C = 20\underline{\mathbf{i}} \qquad \underline{\mathbf{v}}_B = 4.5\underline{\mathbf{j}} \quad (m/s)$

$\underline{\mathbf{r}}_C = 20t\,\underline{\mathbf{i}} \qquad \underline{\mathbf{r}}_B = (60 + 4.5t)\underline{\mathbf{j}}\ (m)$

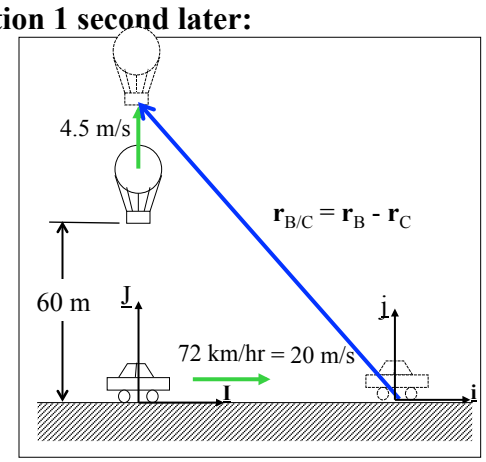
$$r_{B/C} = |\underline{\mathbf{r}}_{B/C}| = \sqrt{\underline{\mathbf{r}}_{B/C} \bullet \underline{\mathbf{r}}_{B/C}}$$

$$r_{B/C}^2 = (20t)^2 + (60 + 4.5t)^2$$

$$2r_{B/C}\,\dot{r}_{B/C} = 2(20t)(20) + 2(60 + 4.5t)4.5$$

Divide through by 2 $r_{B/C}$ & set t = 1

$$\dot{r}_{B/C} = 690.25/67.52 = 10.22(m/s)$$



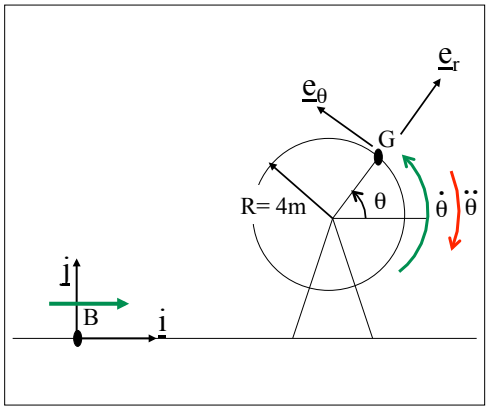
Alternative Method (Vectors!):

$$\dot{r}_{B/C} = \underline{V}_{B/C} \bullet \frac{\underline{r}_{B/C}}{|\underline{r}_{B/C}|} = (\underline{V}_B - \underline{V}_C) \bullet \frac{(\underline{r}_B - \underline{r}_C)}{|\underline{r}_{B/C}|} = (-20,4.5) \bullet \frac{(-20,64.5)}{|67.5|} = \frac{690.3}{67.5} = 10.2\ \text{(m/s)}$$

32

Given: **The ferris wheel rotates at $\dot{\theta}$= 2 r/s, $\ddot{\theta}$ = -1 r/s^2 and the boy (B) walks to the right at a constant speed of 2 m/s.**

Find: **The velocity and acceleration of girl (G) on the ferris wheel relative to boy B** => $\underline{\mathbf{v}}_{G/B} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_G = ?$ $\underline{\mathbf{a}}_{G/B} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_G = ?$



$\underline{e}_r = \cos\theta\ \underline{i} + \sin\theta\ \underline{j}$ or $\underline{i} = \cos\theta\ \underline{e}_r - \sin\theta\ \underline{e}_\theta$

$\underline{e}_\theta = -\sin\theta\ \underline{i} + \cos\theta\ \underline{j}$ $\quad$ $\underline{j} = \sin\theta\ \underline{e}_r + \cos\theta\ \underline{e}_\theta$

Solution:

Defining unit vectors $\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$, $\underline{\mathbf{e}}_r$, & $\underline{\mathbf{e}}_\theta$ shown *velocities* & *accelerations* are then:

$\underline{\mathbf{v}}_B = 2\underline{\mathbf{i}}$ (m/s) $\quad$ $\underline{\mathbf{a}}_B = 0$

$\underline{\mathbf{v}}_G = 4\text{ m }(2\text{ r/s})\ \underline{\mathbf{e}}_\theta = 8\ \underline{\mathbf{e}}_\theta$ (m/s)

$\underline{\mathbf{a}}_G = -4\text{ m }(2\text{ r/s})^2\ \underline{\mathbf{e}}_r + 4\ (-1\text{ r/s}^2)\ \underline{\mathbf{e}}_\theta$
$= -16\ \underline{\mathbf{e}}_r - 4\ \underline{\mathbf{e}}_\theta$ (m/s^2)

Must convert axes to combine vectors:

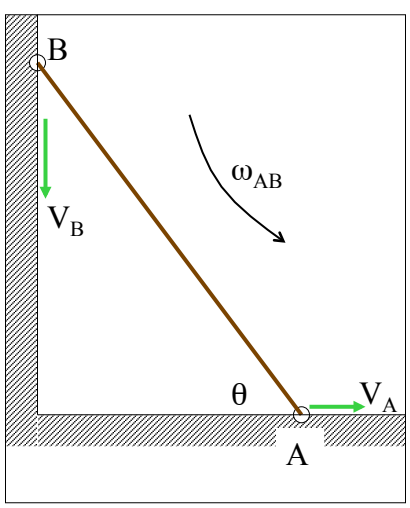
$$\boxed{\begin{aligned}\underline{\mathbf{v}}_{G/B} &= \underline{\mathbf{v}}_G - \underline{\mathbf{v}}_B = 8\ \underline{\mathbf{e}}_\theta - 2\underline{\mathbf{i}} \\ &= -(8\sin\theta + 2)\ \underline{\mathbf{i}} + 8\cos\theta\ \underline{\mathbf{j}} \quad \text{OR} \\ &= -2\cos\theta\ \underline{\mathbf{e}}_r + (2\sin\theta + 8)\ \underline{\mathbf{e}}_\theta \quad \text{(m/s)}\end{aligned}}$$

and since $\underline{\mathbf{a}}_B = 0$:

$$\boxed{\underline{\mathbf{a}}_{G/B} = \underline{\mathbf{a}}_G = -16\ \underline{\mathbf{e}}_r - 4\ \underline{\mathbf{e}}_\theta \text{ (m/s}^2\text{)}}$$

33

Two points on a rigid body:



$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$

$v_B\ \underline{\mathbf{j}} = v_A\ \underline{\mathbf{i}} + \omega_{AB}\ \underline{\mathbf{k}} \times AB\ \underline{\mathbf{u}}_{B/A}$

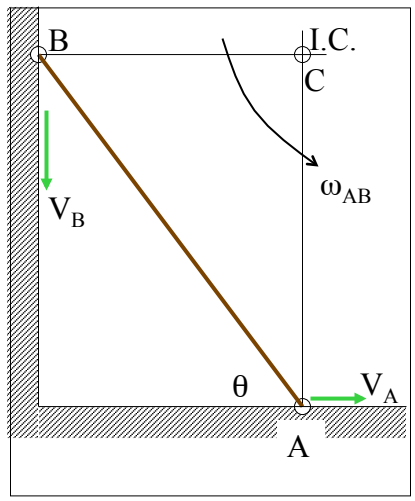
$v_B\ \underline{\mathbf{j}} = v_A\ \underline{\mathbf{i}}$
$\quad - AB\ \omega_{AB}(\sin\theta\ \underline{\mathbf{i}} + \cos\theta\ \underline{\mathbf{j}})$

Equating $\underline{\mathbf{i}}$ & $\underline{\mathbf{j}}$ components:

$\underline{\mathbf{i}} \rightarrow v_A - AB\ \omega_{AB}\sin\theta = 0$

$\underline{\mathbf{j}} \rightarrow v_B = AB\ \omega_{AB}\cos\theta$

$$\frac{v_A}{v_B} = \frac{AB\ \omega_{AB}\sin\theta}{AB\ \omega_{AB}\cos\theta}$$



Using Instant Centers (IC):

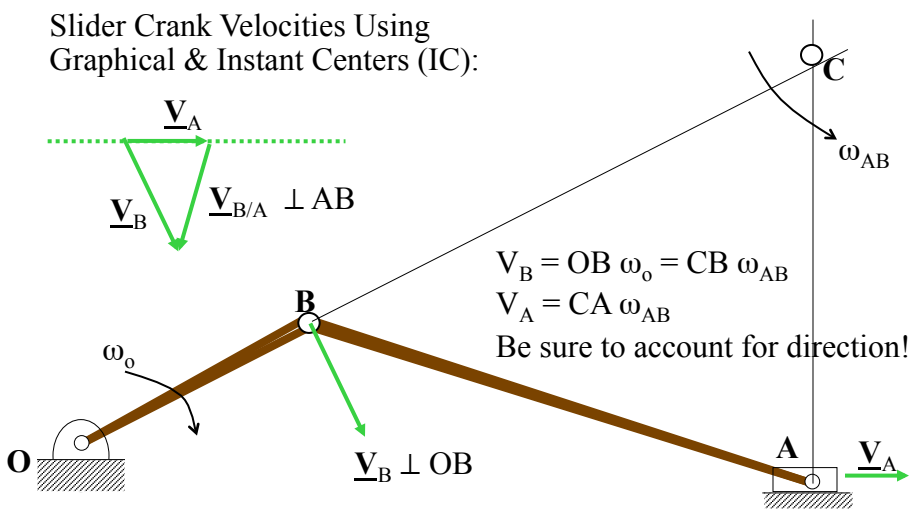
$V_A = AC\ \omega_{AB}\ [\underline{i}]$

$V_B = -BC\ \omega_{AB}\ [\underline{j}]$

$AC = AB \sin\theta$
$BC = AB \cos\theta$

$$\frac{v_A}{v_B} = \frac{AB\ \omega_{AB} \sin\theta}{AB\ \omega_{AB} \cos\theta}$$

Slider Crank Velocities Using Graphical & Instant Centers (IC):

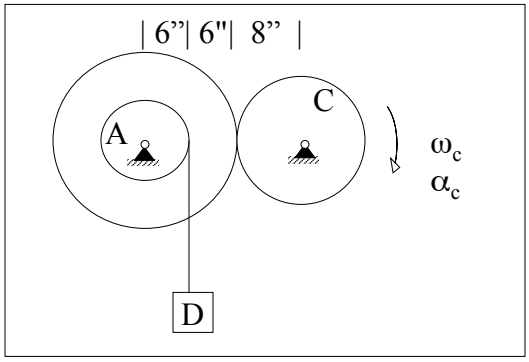


$V_B = OB\ \omega_o = CB\ \omega_{AB}$
$V_A = CA\ \omega_{AB}$
Be sure to account for direction!

$V_A = (OB / CB)\ CA\ \omega_o$

Given: $\omega_c = 2$ r/s

$\alpha_c = 6$ r/s^2



Find: v_D, a_D

$r_A = 6"$ $r_B = 12"$ $r_C = 8"$

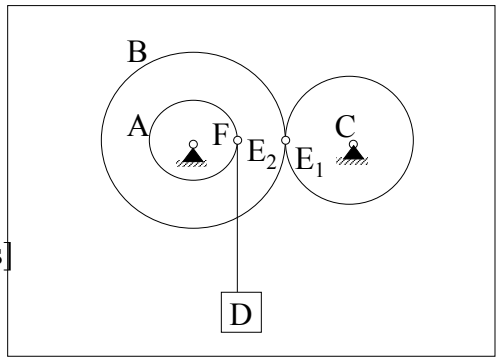
$\omega_C = 2$ r/s $\alpha_C = 6$ r/s^2

$V_{E1} = r_C\omega_C = 8 \bullet 2 = 16$ in/s $\uparrow$

$V_{E2} = 16 \uparrow = r_B\omega_B = 12\ \omega_B$

so: $\omega_B = 4/3 = \omega_A$ [r/s]

$V_D = V_F = \omega_A r_A = 4/3\ (6) = 8 \uparrow$ [in/s]



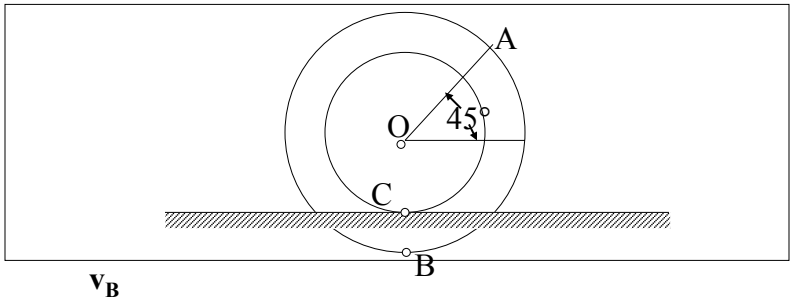
$a_{E1} = \alpha_C r_C \uparrow + \omega_C^2 r_C \rightarrow$

$= 6\ (8) \uparrow + 4(8) \rightarrow = 48 \uparrow + 32 \rightarrow$ [in/s^2]

$a_{E2\,t} = a_{E1t} = 48 \uparrow = \alpha_B r_B = \alpha_B(12)$ $\alpha_B = 4 = \alpha_A$

$a_{F\,t} = \alpha_A r_A = 4(6) = 24 \uparrow = a_D$ [in/s^2]

Given: $r_o = 3'$ $r_i = 2'$ $v_o = 10$ f/s **no slip**



Find: v_B

$\underline{v}_o = 10$ ft/s ➡

$\underline{v}_A = ?$

$v_c = v_o + \omega r = 10 - 2\omega = 0$

$\omega = 5$ ↻ or $-5\,\underline{k}$

$\underline{v}_B = \underline{v}_o + \underline{\omega} \times \underline{r}_{B/o} = 10\,\underline{i} + -5\underline{k} \times -3\underline{j} = -5\,\underline{i}$ [ft/s] ⬅

or $\underline{v}_B = \underline{v}_c + \underline{\omega} \times \underline{r}_{B/c} = 0 + 5\underline{k} \times -1\underline{j} = -5\,\underline{i}$ [ft/s] ⟵

Kinetics Summary

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?
  - i) Newton's Laws

$$\sum \mathbf{F} = m\,\underline{\mathbf{a}}_{CG} \qquad \sum \underline{\mathbf{M}}_P = I_{CG}\,\alpha + r_{eff}\,m a_{CG}$$

  - ii) Work- Energy & Conservation of Energy

$$U_{A-B} = \int_{s_A}^{s_B} m a_t\,ds = \int_{v_A}^{v_B} m v\,dv = \frac{1}{2} m\left(v_B^2 - v_A^2\right) = \Delta T_{A-B}$$

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

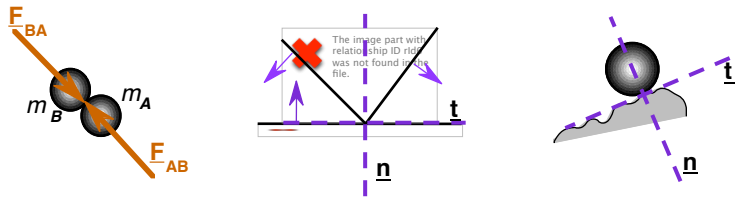
  - iii) Impulse - Momentum & Conservation of Momentum

$$\underline{\mathbf{I}} = \int \underline{\mathbf{F}}_R\,dt = \int d\underline{\mathbf{L}} = \Delta\underline{\mathbf{L}}$$

  - Typical forces
    - Springs $F = k\,(s - s_0)$
    - Friction $\mathbf{F}_f = \mu_{s/k}\,\mathbf{N}$
    - Gravitation $\underline{\mathbf{F}} = m\,\underline{\mathbf{g}}$

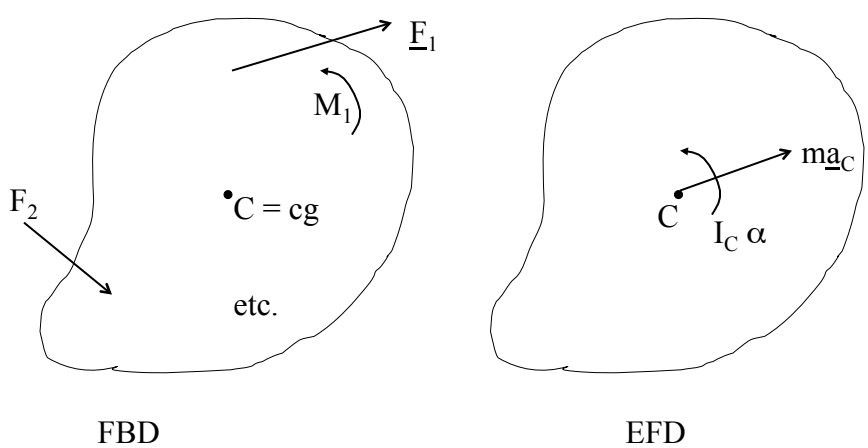
Particle Kinetics: Free Body Diagrams

- Free Body Diagrams:
  - Isolate the particle/system of interest (i.e. boundaries)
  - For noting action-reaction between particles/bodies it is important to identify the **common normal-tangent @ the point of contact** (often one or the other is easily identified)



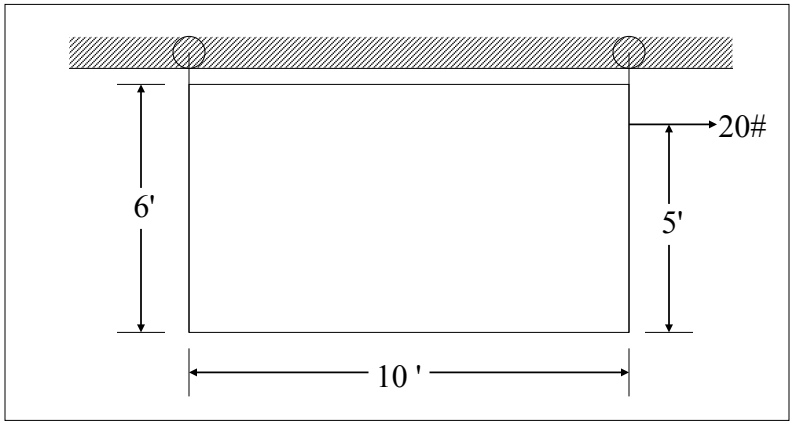
  - Include ALL forces (& later => moments)
    - Field forces (gravity, electro-magnetic fields etc)
    - Viscous forces (aerodynamic drag, fluid flows, etc)
    - Contact forces (touching elements) -- Most common
  - For motion over an interval --- draw in a general position!

Kinetics: $\Sigma \underline{F} = m\underline{a}_C$ $\Sigma M_C = I_C \alpha$

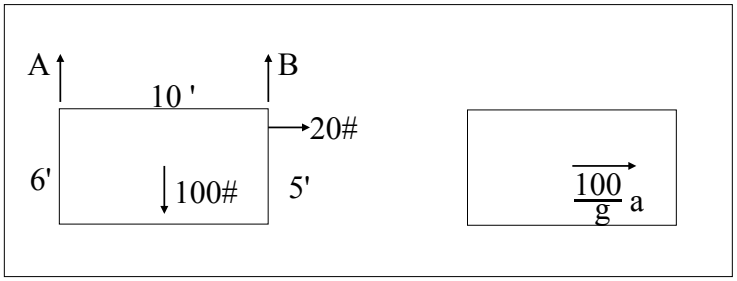


$$\sum \underline{\mathbf{M}}_P = I_C \alpha + r_{eff} m a_C$$

Given: the 20# force is applied to the sliding door which weighs 100 #
Find: the reactions at the frictionless roller supports



Find: the reactions at the frictionless roller supports



$\Sigma F \rightarrow = 20 = (100/g)\, a \longrightarrow a = g/5$

$\Sigma F \uparrow = A + B - 100 = 0$ => $B = 54$ #

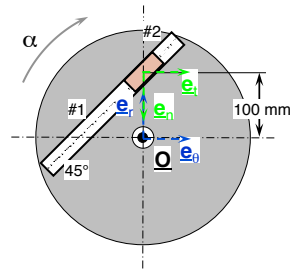
$\Sigma M_{CG} \circlearrowleft = 5B - 5A - (20)\, 2 = 0$ $A = 46$ #

$\Sigma M_A \circlearrowleft = 10B - 5(100) + (20) = 3\,(100/g)(g/5)$

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

Given:

- The slider (m=2 kg) fits loosely in the smooth slot of the disk which lies in a horizontal plane and rotates about a vertical axis through point **O**.
- The slider is free to move only slightly along the slot in either direction before one (but not both) of the two wires #1 or #2 becomes taut.
- The disk starts from rest at time t = 0 and has a constant clockwise angular acceleration of α=0.5 r/s^2.

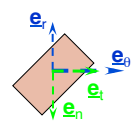


Find:

(A) Determine the **TENSION** ($\mathbf{T}_2$) in wire #2 at t =1 second

(B) Determine the **REACTION FORCE** (**N**) between the slot and the block, again at t =1 second.

(C) Determine the **TIME** (t) at which the tension in wire #2 goes slack and wire #1 becomes taut.



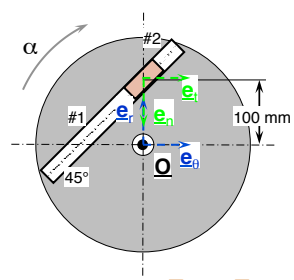
Solution:

- Asks for FORCES (**T**,**N**) so we must first establish kinematics (accelerations!)
- "Move only slightly" means it is effectively fixed relative to the slot/disk, thus
- The slider travels a circle about **O** & path ($\underline{e}_n$-$\underline{e}_t$) axes
  or polar ($\underline{e}_r$-$\underline{e}_\theta$) axes are convenient

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

Solution (continued):

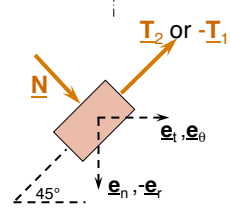
- Construct FBD
- Use disk kinematics (α=0.5 r/s^2 CW constant) to determine slider's total acceleration



$$\rho = 0.100m = r \quad \Rightarrow constant$$

$$\therefore \quad \dot{\rho} = \ddot{\rho} = \dot{r} = \ddot{r} = 0$$

- Not instantaneous - integrate angular acceleration



$$\int_0^\omega d\omega = \int_0^t \alpha\, dt = \int_0^t 0.5\, dt$$

$$\omega = 0.5t$$

$$\underline{\mathbf{a}}_s = \alpha r \underline{\mathbf{e}}_\theta - r\omega^2 \underline{\mathbf{e}}_r$$

$$= \dot{v}\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\underline{\mathbf{e}}_n = \alpha r \underline{\mathbf{e}}_t + \omega^2 r \underline{\mathbf{e}}_n$$

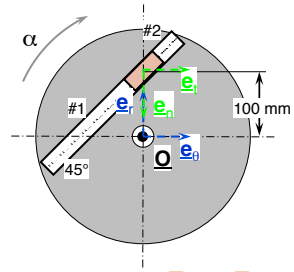
Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

Solution (continued):

- Newton's Law can be applied along ANY two independent directions to resolve unknown reactions
  - Sum force components along (n-t, r-θ)

$$\mathbf{T}_2 \cos 45 + \mathbf{N} \sin 45 = m\alpha r$$

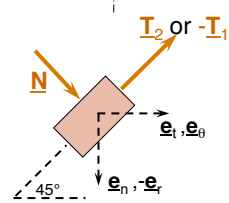
$$\mathbf{T}_2 \sin 45 - \mathbf{N} \cos 45 = -m\omega^2 r$$



  - OR to simplify algebra of unknowns, choose the directions along the unknown reactions and sum both *forces* and *acceleration components*

$$\mathbf{T} = m\left(\alpha r \cos 45 - \omega^2 r \sin 45\right) = \frac{mr\sqrt{2}}{2}\left(\alpha - \omega^2\right)$$

$$\mathbf{N} = m\left(\alpha r \cos 45 + \omega^2 r \sin 45\right) = \frac{mr\sqrt{2}}{2}\left(\alpha + \omega^2\right)$$



  - *ASIDE*: This IS the geometric equivalent to simultaneously solving the first set of constraints to yield expressions for the unknowns
  - Noting the similarity of the expressions (± : + for **N**, - for **T**)

$$\mathbf{N},\mathbf{T}_2 = \frac{mr\sqrt{2}}{2}\left(\alpha \pm \omega^2\right)$$

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

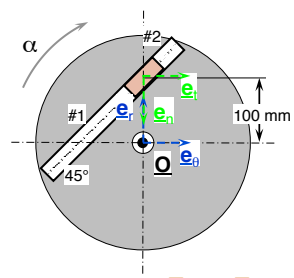
Solution (continued):

- Substituting the known expressions for α & ω(*t*)

$$\mathbf{N},\mathbf{T}_2 = \frac{mr\sqrt{2}}{2}\left(\alpha \pm \omega^2\right)$$

$$= \frac{2\,kg * 0.1\,m * \sqrt{2}}{2}\left\{0.5 \pm (0.5t)^2\right\}(r/s^2)$$

$$\mathbf{N},\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{1 \pm 0.5t^2\right\}(N)$$



(A) So for t=1, the TENSION $\underline{\mathbf{T}}_2$ is

$$\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{1 - 0.5(1)^2\right\}(N) = \underline{\underline{\frac{\sqrt{2}}{40}(N) = 0.035(N)}}$$

(B) At t=1, the NORMAL REACTION $\underline{\mathbf{N}}$ is

$$\mathbf{N} = \frac{\sqrt{2}}{20}\left\{1 + 0.5(1)^2\right\}(N) = \underline{\underline{\frac{3\sqrt{2}}{40}(N) = 0.106(N)}}$$

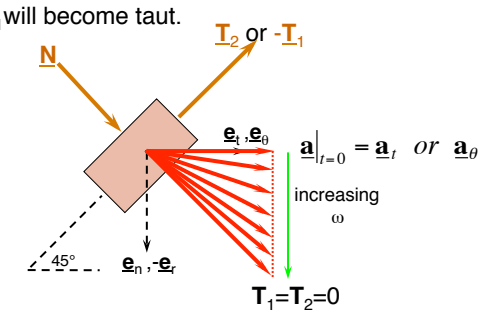
(C) The time when TENSION $\underline{\mathbf{T}}_2$ goes to zero is

$$\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{1 - 0.5t^2\right\}(N) = 0 \Rightarrow 1 - 0.5t^2 = 0 \Rightarrow t = \sqrt{2} \Rightarrow \underline{\underline{t = 1.414\,(s)}}$$

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

Langiappe:

- The acceleration vector starts off completely in the lateral (θ or t) direction here (ω=0). Since cables/wires/ropes cannot PUSH, only $\mathbf{T}_2$ can be engaged in balancing the (r or n) component of the side wall reaction **N**
- The tangential acceleration component remains constant
- As the disk speeds up (ω >0), the normal component increases
- When the total acceleration vector aligns with the normal reaction force between the block & slot, the cord/wire tensions are both zero momentarily, and as $\mathbf{T}_2$ goes slack, $\mathbf{T}_1$ will become taut.



Impulse / Momentum

$$\int_{t_1}^{t_2} \underline{F}\, dt = \int m\underline{a}dt$$
$$= \int m d\underline{v}$$
$$= m\underline{v}_2 - m\underline{v}_1$$

If mutual forces cancel, impulses cancel
(not true of work)

coef. Of restitution

$$e = \frac{\text{rel. norm. sep. vel.}}{\text{rel. norm. app. vel.}}$$

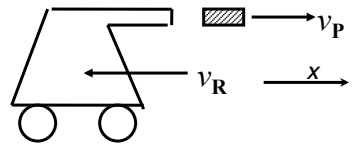
Angular Momentum:

$$\int \underline{M}c_o dt = \int Ic_o \alpha\, dt$$
$$= \int Ic_o d\omega$$
$$= \int Ic_o\omega_2 - Ic_o\omega_1$$

Example: Conservation of Momentum

Given:

- An artillery gun (m_G) fires a shell (m_P) with a speed v_p



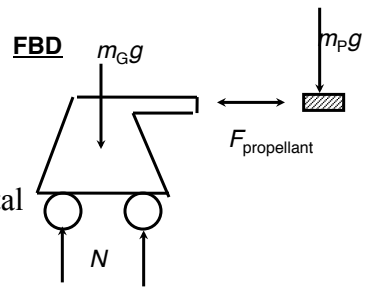
Find:

(A) The recoil speed (v_R) of the gun

Solution:

- FBD of system components, just as shell leaves the gun
- Rectilinear motion (i.e. only horizontal motion of interest here)
- Propellant firing is internal to the system
  - System momentum is conserved in the horizontal direction $\Delta \mathbf{L}_{\mathbf{sys}} = 0$

$$\Delta L_{x-system} = m_G(v_G - 0) + m_P(v_P - 0) = 0$$

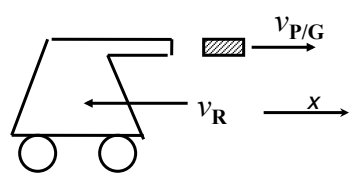


$$v_R = -v_G = \frac{m_P}{m_G} v_p$$

Example: Conservation of Momentum

Given:

- More often, a "muzzle velocity" ($v_{P/G}$)or speed of the shell relative to the gun barrel is specified

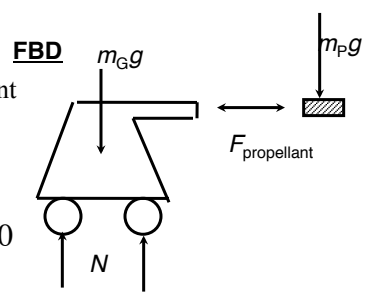


Find:

(A) The recoil speed (v_R) of the gun

Solution:

- FBD (same), Rectilinear motion & Propellant firing is internal



$$\Delta \mathbf{L}_{\mathbf{sys}} = 0$$

$$\Delta L_{x-system} = m_G(v_G - 0) + m_P(v_P - 0) = 0$$

$$v_P = v_G + v_{P/G}$$

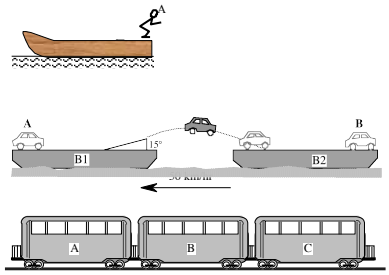
$$m_G v_G + m_P(v_G + v_{P/G}) = 0$$

$$v_R = -v_G = \left(\frac{m_P}{m_G + m_P}\right) v_{P/G}$$

Example: Conservation of Momentum

Given:

- Numerous examples with similar circumstances, rephrasing the wording
  - Kid(s) on a boat in still water, one jumps off
  - Car lands on a barge & skids to rest relative to barge
  - Rail cars collide & stay attached



Find:

(A) The resulting speeds of each element

(B) A time it takes to "skid to rest"

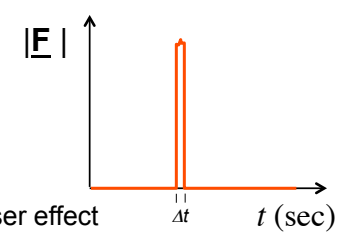
Solution:

- Similar conservation of momentum relations

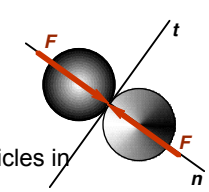
$$\Delta \mathbf{L}_{\mathbf{sys}} = 0$$

Particle Kinetics: Impulse-Momentum

- <u>Impact Problems</u>:
  - Reformulation of one type of Impulse-Momentum $\underline{\mathbf{L}} = m\underline{\mathbf{v}}$
  - Impulsive Forces characterized by
    - LARGE MAGNITUDE
    - SHORT TIME DURATION
    - Ex: explosions, ball-bat, club-golf ball

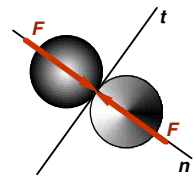


  - Neglect other conventional forces of lesser effect for the short time interval
    - Springs
    - Gravity
    - Many Reaction forces (BUT NOT ALL!)
  - Good opportunity to look at the SYSTEM of particles in simplifying the problem (reactions are internal!)



Particle Kinetics: Impulse-Momentum/ Impact

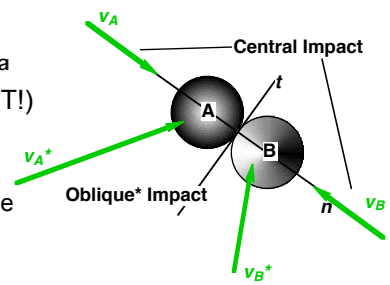
- Impact
  - Locate Common Normal/Tangent
    - Line of contact/impact - the NORMAL!
  - Forces of interaction
    - Equal, Opposite, Co-linear
  - Very complex internal phenomena, captured by *Coefficient of Restitution*



$$e = \left.\frac{(V_{relative}\ \text{Separation})}{(V_{relative}\ \text{Approach})}\right|_{\text{Common Norma}}$$

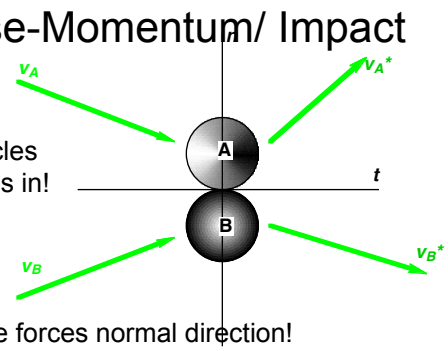
(good derivation in B&J text --- READ IT!)

  - *Central & Oblique Impacts*
    - Velocities are NOT co-linear with the line of impact (i.e. the common normal)



Particle Kinetics: Impulse-Momentum/ Impact

- Solving Impact Problems !



(1) Tangential Direction: individual particles have no net external impulsive forces in!

$$m_A v_{At} = m_A v_{At}^* \quad \& \quad m_B v_{Bt} = m_B v_{Bt}^*$$

(2) System of particles: No net impulsive forces normal direction!

$$\Delta \underline{L}_n\Big|_{SYS} = 0 \Rightarrow m_A v_{An} + m_B v_{Bn} = m_A v_{An}^* + m_B v_{Bn}^*$$

(3) Coefficient of Restitution: Rel. Velocities along Common NORMAL!

$$e = \left.\frac{v_{Relative\ Separation}}{v_{Relative\ Approach}}\right|_{\text{Normal}} = \frac{v_{Bn}^* - v_{An}^*}{v_{An} - v_{Bn}}$$

(Perfectly Plastic) $0 \le e \le 1$ (Perfectly Elastic)

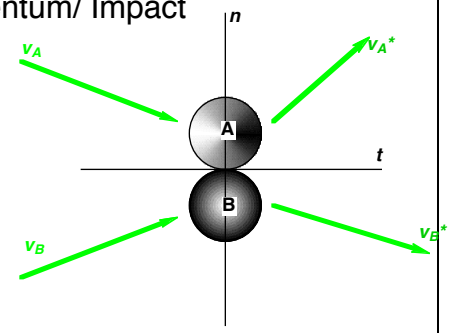
Particle Kinetics: Impulse-Momentum/ Impact

- Solving constraint relations !

$$v_{At} = v_{At}^{*} \;\; \& \;\; v_{Bt} = v_{Bt}^{*}$$

$$(1) \;\; v_{Bn}^{*} = v_{Bn} + \frac{m_A}{m_B}\left(v_{An} - v_{An}^{*}\right)$$

$$(2) \;\; v_{Bn}^{*} = e\left(v_{An} - v_{Bn}\right) + v_{An}^{*}$$



- From which the unknown rebound (normal) component of velocities become

$$(1) \;\; v_{An}^{*} = \left(\frac{m_A - m_B e}{m_A + m_B}\right) v_{An} + \left(\frac{m_B}{m_A + m_B}\right)(1+e) v_{Bn}$$

$$(2) \;\; v_{Bn}^{*} = \left(\frac{m_A}{m_A + m_B}\right)(1+e) v_{An} + \left(\frac{m_B - m_A e}{m_A + m_B}\right) v_{Bn}$$

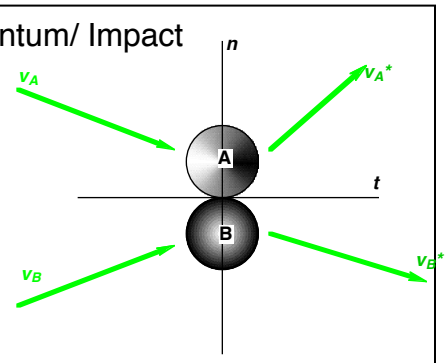
Particle Kinetics: Impulse-Momentum/ Impact

- What if $m_A >>> m_B$?

$$v_{At} = v_{At}^{*} \;\; \& \;\; v_{Bt} = v_{Bt}^{*}$$

$$(1) \;\; v_{Bn}^{*} = v_{Bn} + \underbrace{\frac{m_A}{m_B}}_{0}\left(v_{An} - v_{An}^{*}\right)$$

$$(2) \;\; v_{Bn}^{*} = e\left(v_{An} - v_{Bn}\right) + v_{An}^{*}$$



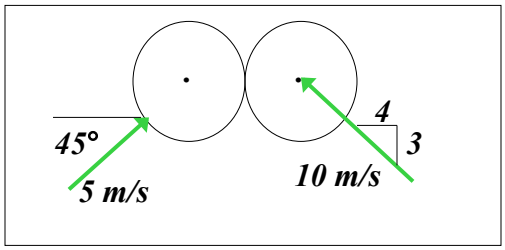
- From which the unknown rebound (normal) component of velocities become

$$(1) \;\; v_{An}^{*} = -e\, v_{An} + (1+e) v_{Bn}$$

$$(2) \;\; v_{Bn}^{*} = v_{Bn}$$

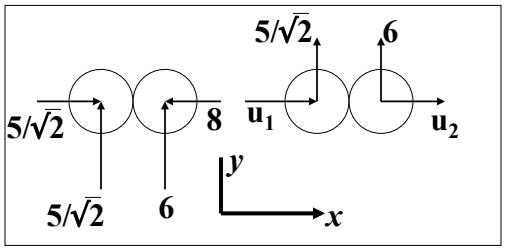
Given: two balls of equal mass with the velocities shown collide, coefficient of rest = 0.8

Find: velocities after impact



x - mom.: $u_1 + u_2 = 5/\sqrt{2} - 8$

Rest: $u_2 - u_1 = 0.8\,[(5/\sqrt{2}) + 8\,]$



$u_2 = 2.76 \qquad u_1 = -7.23$

Work / Energy :

$$T_1 + V_1 + W_{NC} = T_2 + V_2$$

$$\text{where} \quad T = \frac{1}{2} m\, v_c^2 = \frac{1}{2} I_c\, \omega^2 = \frac{1}{2} I_c\, \omega^2$$

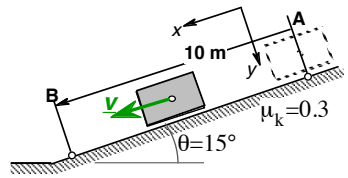
$$V_{grav} = -mgh$$

$$V_{spring} = \frac{1}{2} k\, x^2 \qquad \text{where } x = l - l_o = \text{stretch}$$

Work-Energy Example ref ~Meriam & Kraige 3/11

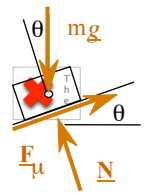
Given:

- A crate of mass m slides down an incline
- m =50 kg, θ=15°, μ_k=0.3,
- Reaches **A** with speed 4 m/s



Find:

(A) Speed of crate v_B as it reaches a point **B** 10 m down the incline from **A**



Solution:

- Rectilinear motion, align axes accordingly - i.e. || & ⊥ to incline
- FBD of crate in general position (working over a motion interval here)
- No movement ⊥ to incline so

$$\sum F_y = mg\cos\theta - N = 0 \quad \Rightarrow \quad N = mg\cos\theta$$

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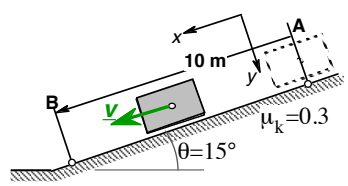
Work-Energy Example ref ~Meriam & Kraige 3/11

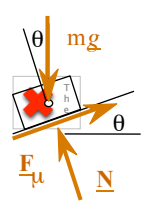
Solution (cont'd):

- Work done is due to the resultant forces *in direction of displacement* (i.e. down incline) & includes Friction & component of Weight



$$U_{A\text{-}B} = (mg\sin\theta - N\mu_k)\Delta x_{AB}$$
$$= (mg\sin\theta - mg\cos\theta\mu_k)\Delta x_{AB}$$

- Principle of Work-Energy then says



$$U_{A\text{-}B} = \Delta T_{A\text{-}B} = T_B - T_A$$

$$\Rightarrow \quad T_B = U_{A\text{-}B} + T_A$$

$$\frac{1}{2}mv_B^2 = mg(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + \frac{1}{2}mv_A^2$$

$$v_B = \sqrt{2g(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + v_A^2}$$

$$v_B = \sqrt{2*9.81(m/s^2)*(\sin 15° - \cos 15°*0.3)*10m + (4m/s)^2}$$

$$v_B = \underline{\underline{3.15\ \ m/s}}$$

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Work-Energy: Example ref ~Meriam & Kraige 3/13

Given:

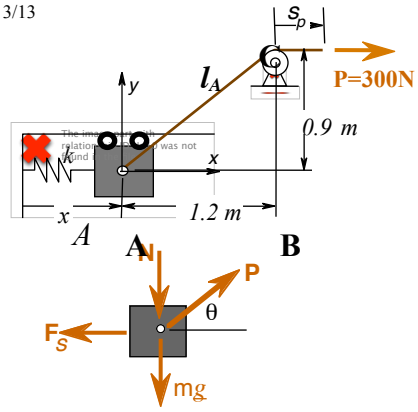
- Block (m =50 kg) mounted on rollers
- Massless spring w/ k=80 N/m
- Released from rest at **A** where spring has initial stretch of 0.233 m
- Cord w/ constant tension **P**=300 N attaches to block & routed over frictionless/ massless (ideal) pulley @ **C**

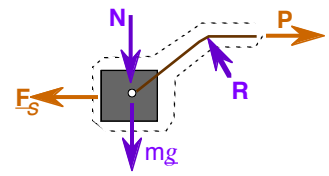
Find:

(A) Speed of block v_B as it reaches a point **B** directly under the pulley.

Solution:

- Again, rectilinear motion, align axes accordingly
- FBD of block in general position (working over a motion interval here)
- Look at alternative - include the rope in as part of the SYSTEM - reduce FDB to an ***ACTIVE Force Diagram!***





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Work-Energy: Example ref ~Meriam & Kraige 3/13

Solution (cont'd):

ACTIVE Force Diagram!

- Eliminate Normal Forces ⊥ to displacement @ their point of contact {THEY DO NO WORK!}
  - Weight (**mg**) & Roller reactions (**N**)
  - Pulley force on rope (**R**)
- Active forces DO work on the system
  - Spring Force (F_s) => opposes motion

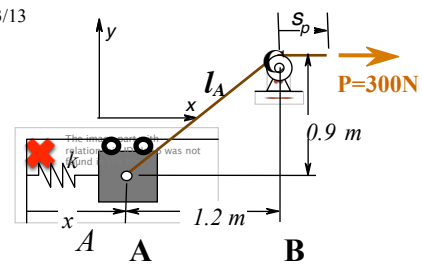
$$F_s = -kx$$

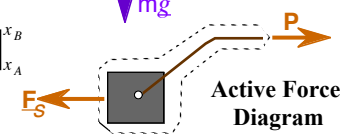
$$U_{AB_{F_s}} = \int_{x_A}^{x_B} F_s dx = \int_{x_A}^{x_B} -kx dx = \frac{1}{2}kx^2\Big|_{x_A}^{x_B}$$

$$= -\frac{1}{2}k(x_B^2 - x_A^2)$$

  - Assuming block can actually reach B

$$U_{AB_{F_s}} = -\frac{1}{2}80(N/m)\{(1.2+0.233)^2 - 0.233^2\}(m^2) = -80\,Joules$$





Active Force Diagram

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Work-Energy: Example ref ~Meriam & Kraige 3/13

Solution (cont'd):

- Calculate Work done on system
  - Cord Tension (P) => *constant*
  - Displacement of P

$$L_{cord} = s_P + l = constant$$

$$\Delta s_P = -\Delta l = l_A - l_B$$

$$= \sqrt{1.2^2 + 0.9^2} - 0.9 \cong 0.61m$$

$$U_{AB_P} = P\Delta s = 300(n) * 0.61(m)$$

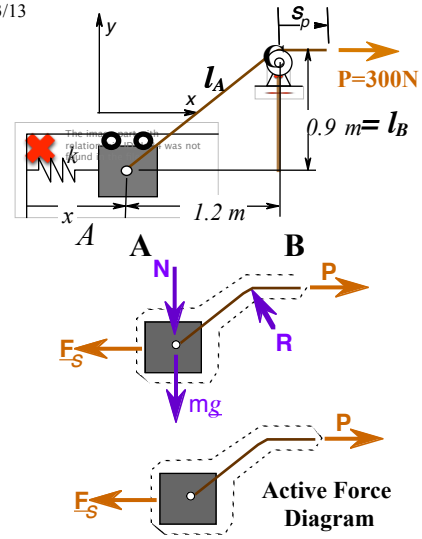
$$= 180 Joules$$

- Work-Energy

$$U_{TOT} = \Delta T_{A-B} = T_B - T_A$$

$$-80 + 180(Joules) = \frac{1}{2} m(v_B^2 - 0)$$

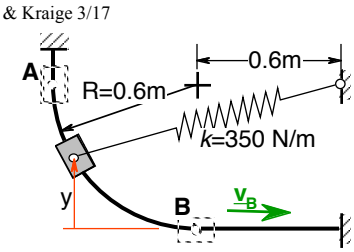
$$\Rightarrow v_B = \sqrt{\frac{100(Joules) * 2}{50 Kg}} = \underline{\underline{2.0 m/s}}$$



Conservation-Energy Example ref ~Meriam & Kraige 3/17

Given:

- m =3 kg slider on circular track shown
- Starting from A with v_A=0
- l_o=0.6 m (unstretched), k=350 N/m
- μ=0 (i.e friction is negligible)

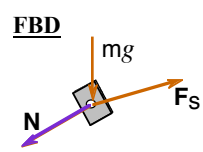


Find:

(A) Velocity of slider as it passes **B**

Solution:

- FBD of crate in general position (working over a motion interval here)



- Identify
  - Conservative Forces

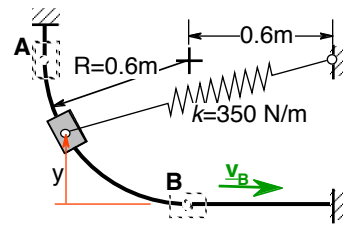
    mg (Weight/Gravity) & **F_s** (Spring)
  - Non-working Constraint Forces

    N (Track reaction force)

Conservation-Energy Example ref ~Meriam & Kraige 3/17

Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!

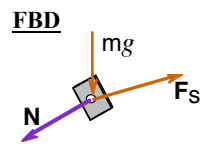


$$\Delta E_{TOT} = \Delta T + \Delta V_g + \Delta V_e = 0$$

$$\Delta T_{AB} = \frac{1}{2}m(v_B^2 - v_A^2) = \frac{1}{2}m(v_B^2 - 0)$$

$$\Delta V_{AB_g} = mg(y_B - y_A) = mg(0 - R)$$

$$\Delta V_{AB_e} = \frac{1}{2}k\left\{(l_B - l_0)^2 - (l_A - l_0)^2\right\}$$



- Pulling together all components & isolating v_B

$$v_B = \sqrt{2gR + \frac{k}{m}\left\{R^2 - (\sqrt{2}R - R)^2\right\}}$$

- Incorporating numerical values of all terms

$$v_B = \sqrt{2*9.81(m/s^2)(0.6m) + \frac{350N/m}{3kg}\left\{(0.6m)^2 - (\sqrt{2}*0.6m - 0.6m)^2\right\}} = \underline{\underline{6.82\ \ m/s}}$$

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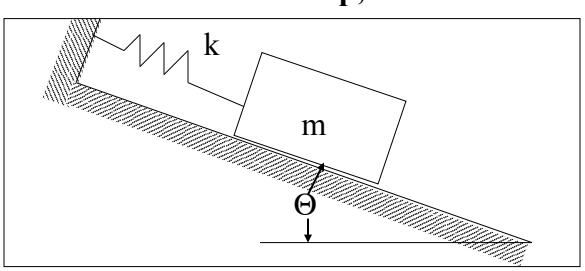
Work/ Energy $\mathbf{T_1 + V_1 + W_{NC} = T_2 = V_2}$

Given: **k, m, Θ**
block released from rest with spring compressed δ

Find: **distance travelled to stop, d**



$T_1 = 0$ $\quad V_1 = {}^1/_2\, k\, \delta^2$

$W_{NC} = -Fd$

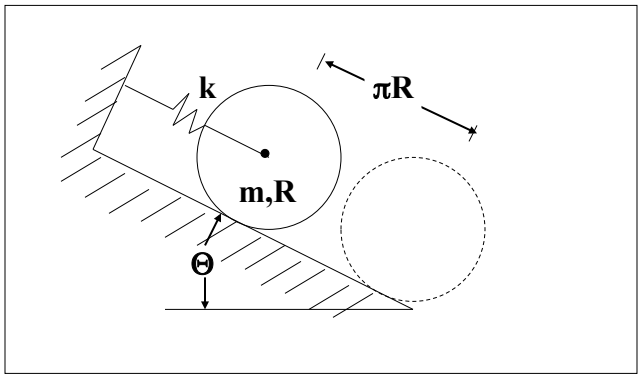
where $N = mg\cos\Theta$, $F = \mu_k N$ <u>while sliding</u>

$T_2 = 0$ $\quad V_2 = {}^1/_2 k(d - \delta)^2 - mgd\sin\Theta$

Given: k, m, R, Θ, I

released from rest, no slip, no initial deflection in spring

Find: speed after 1/2 revolution



$T_1 = 0$
$V_1 = 0$
$W_{NC} = 0$

$T_2 = 1/2\ m\ v_2^2 + 1/2\ I\ \omega_2^2$, $V_2 = 1/2\ k\ (\pi R)^2 - mg\ \pi R \sin\Theta$

need also $v_2 = R\ \omega_2$ (no slip condition)

$0 = 1/2\ m\ v_2^2 + 1/2\ I\ (v_2/R)^2 + 1/2\ k\ (\pi R)^2 - mg\ \pi R \sin\Theta$

Kinetics Summary

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?
  - i) Newton's Laws

$$\sum \mathbf{F} = m\ \underline{\mathbf{a}}_{CG} \qquad \sum \underline{\mathbf{M}}_P = I_{CG}\ \alpha + r_{eff} m a_{CG}$$

  - ii) Work- Energy & Conservation of Energy

$$U_{A-B} = \int_{s_A}^{s_B} m a_t ds = \int_{v_A}^{v_B} m v dv = \frac{1}{2} m\left(v_B^2 - v_A^2\right) = \Delta T_{A-B}$$

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

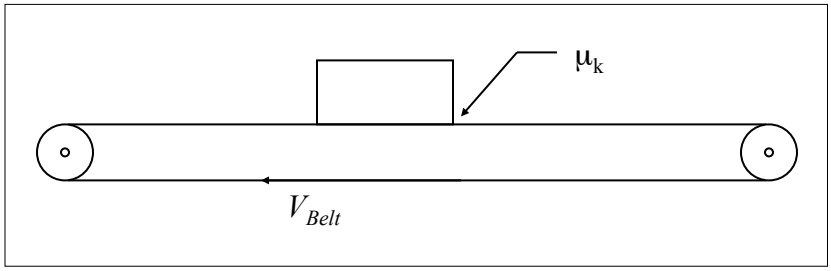
  - iii) Impulse - Momentum & Conservation of Momentum

$$\underline{\mathbf{I}} = \int \underline{\mathbf{F}}_R dt = \int d\underline{\mathbf{L}} = \Delta \underline{\mathbf{L}}$$

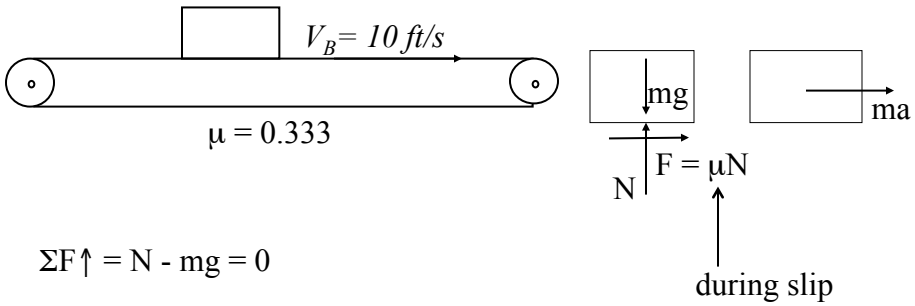
  - Typical forces
    - Springs $F = k\ (s - s_0)$
    - Friction $\mathbf{F}_f = \mu_{s/k}\ \mathbf{N}$
    - Gravitation $\underline{\mathbf{F}} = m\underline{\mathbf{g}}$

Given: Box placed on conveyor with zero initial velocity.

Find: Time during which slip occurs



Find: Time during which slip occurs



$\Sigma F\uparrow = N - mg = 0$

$\Sigma F\rightarrow = \mu N = ma$

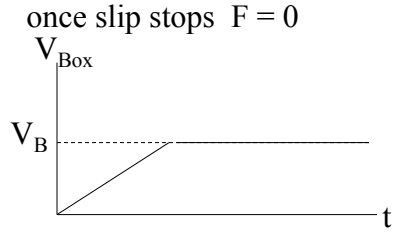
$\mu mg = ma \Rightarrow a = \mu g$

$v = \mu g t$

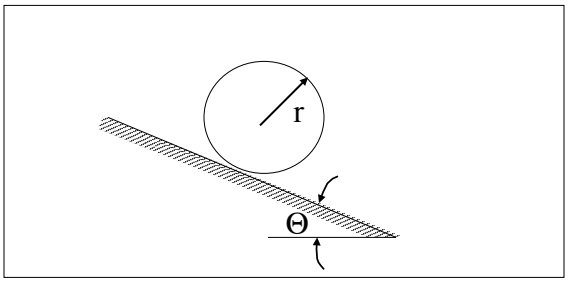
slip stops where $v = v_B$

so $t = v_B / \mu g$

once slip stops F = 0



Given: ball of radius r released from rest on incline, no slip, $\Theta = 30°$

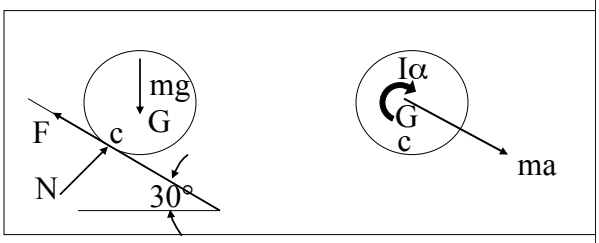


Find: acceleration

$\Sigma F_{\nearrow} = N - mg \cos 30 = 0$

$\Sigma F_{\searrow} = mg \sin 30 - F = ma$

$\Sigma M_{G\circlearrowright} = rF = 2/5\ mr^2\alpha$



if no slip, $a = r\alpha$ so ➡ $mg \sin 30 - 2/5\ mr\alpha = mr\alpha$

so ➡ $\alpha = (5/7)(g/r)(1/2)$; $a = (5g/7)(1/2) = 5g/14$

$$v = a\,t = \frac{5gt}{14}$$

also: $N = \frac{\sqrt{3}}{2}\ mg \rightarrow F_{max} = \mu_s N = \frac{\sqrt{3}\ \mu\ mg}{2}$

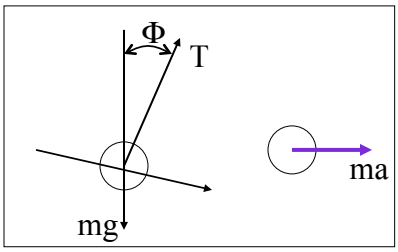
$$F = \frac{2}{5}\ mr\alpha = \frac{mg}{7}$$

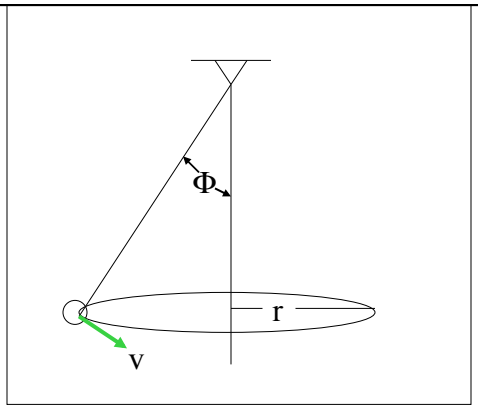
$F \leqslant F_{max}$ requires $\frac{1}{7} \leqslant \frac{\sqrt{3}\ \mu}{2}$

or $\mu \geqslant \frac{2}{7\sqrt{3}}$

Given: given v , r

Find: Φ; tension





$ma = m\, r\, \omega^2$

$ma = m\, v^2/r$

$\Sigma F \uparrow = T \cos \Phi - mg = 0$

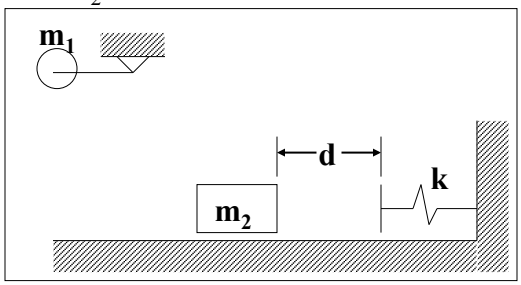
$\Sigma F \rightarrow = T \sin \Phi = m\, v^2/r$

<u>or</u> $\Sigma F \searrow = mg \sin \Phi = ma \cos \Phi$

so $\tan \Phi = \frac{a}{g} = \frac{v^2}{gr}$

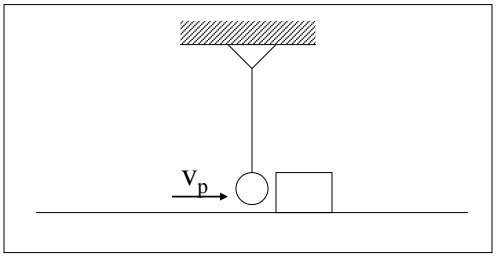
Given: m_1 released from rest, strikes m_2

Find: max spring compression

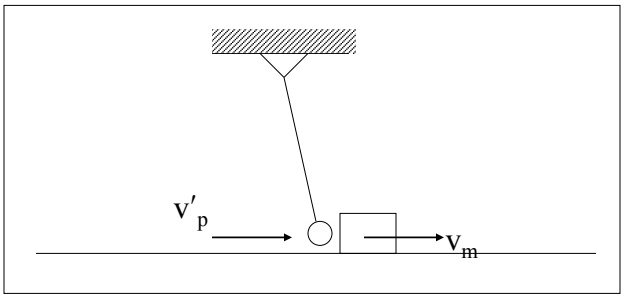


State ① as shown

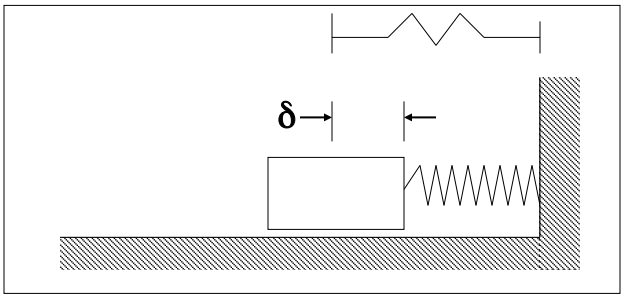
State ②



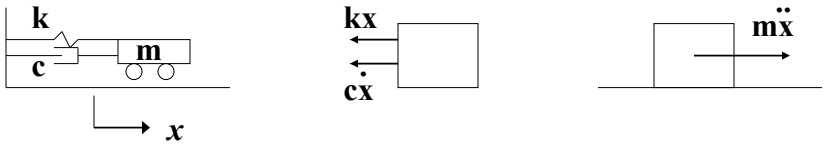
State ③



State ④



Vibrations



$\Sigma F \rightarrow = -kx - c\dot{x} = m\ddot{x}$

Or $m\ddot{x} + c\dot{x} + kx = 0$

$\ddot{x} + c\dot{x} / m + kx / m = 0$

general $\ddot{x} + 2\zeta\omega x + \omega^2 x = 0$

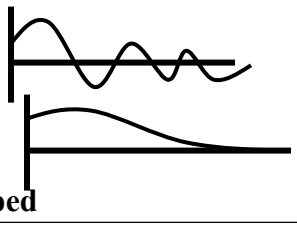
where w = natural freq.

ζ = damping factor

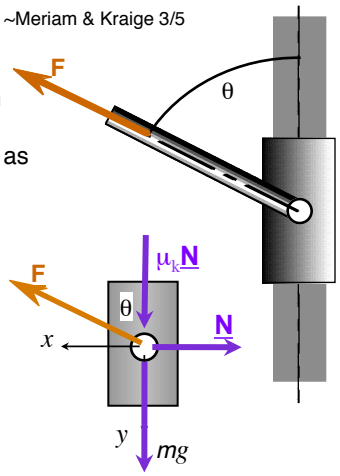
$\zeta < 1$ underdamped

$\zeta > 1$ overdamped

$\zeta = 1$ critically damped



Particle Kinetics: Cartesian Example ref ~Meriam & Kraige 3/5



Given:

- A collar of mass m slides vertically on a shaft with kinetic coefficient of friction, μ_k.
- Applied force **F** is constant but its direction varies as

$$\theta = k\,t \quad k => constant$$

- Collar starts from rest @ θ=0°

Find:

(A) Magnitude of **F** which results in collar coming to rest at θ=90°.

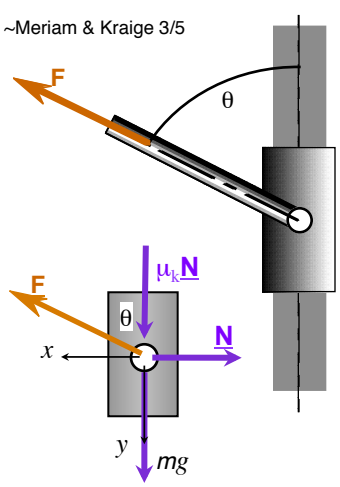
Solution:

- Rectilinear motion! Constrained vertically so align axes accordingly
- FBD of collar in general position (working over a motion interval here)
- Newton's Laws

$$\sum F_x = F\sin\theta - N = 0\ ! \qquad \Rightarrow N = F\sin\theta$$

$$\sum F_y = -F\cos\theta + mg + N\mu_k = ma_y \qquad \& \quad \frac{F}{m}(\sin\theta\mu_k - \cos\theta) + g = a_y$$

Particle Kinetics: Cartesian Example ref ~Meriam & Kraige 3/5



Solution (Cont'd):

$$\frac{F}{m}(\sin\theta\mu_k - \cos\theta) + g = a_y = \frac{dv_y}{dt}$$

- Don't know **F** but DO know it's constant!
- Angle-time relation ($\theta = k\,t$) cleans up
  - Kinematic relationship variables &
  - Proper (Pos,Vel)BC's for integration
    - Starts: v_{yi}=0, θ=0 => t=0
    - Ends: v_{yf}=0, θ=π/2 => t=π/(2k)
  - Start w/ general upper limits

$$\int_0^{v_{yf}} dv_y = \int_0^t \{\frac{F}{m}(\sin(kt)\mu_k - \cos(kt)) + g\}dt$$

$$v_{yf} = \left\{\frac{F}{mk}(-\cos(kt)\mu_k - \sin(kt)) + gt\right\}\Bigg|_0^t$$

$$v_{yf} = \frac{F}{mk}\{[1-\cos(kt)]\mu_k - \sin(kt)\} + gt$$

Particle Kinetics: Cartesian Example ref ~Meriam & Kraige 3/5

Solution (continued):

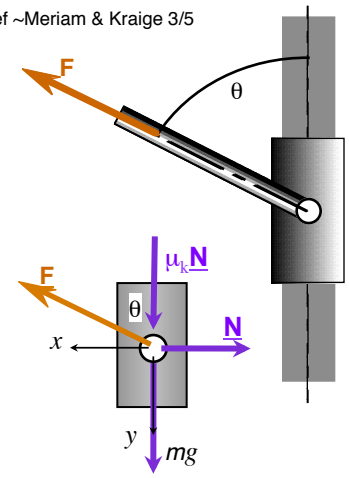
- Now using the final BC for $t=\pi/2k$, $v_{yf}=0$

$$v_{yf} = \frac{\mathrm{F}}{\mathrm{mk}}\{[1-\cos(kt)]\mu_k - \sin(kt)\} + gt$$

$$0 = \frac{\mathrm{F}}{\mathrm{mk}}\{[1-\cos(\pi/2)]\mu_k - \sin(\pi/2)\} + \frac{g\pi}{2k}$$

$$0 = \frac{\mathrm{F}}{\mathrm{mk}}\{\mu_k - 1\} + \frac{g\pi}{2k}$$

$$\therefore \mathrm{F} = \frac{\pi\ \mathrm{m}g}{2(1-\mu_k)}$$



Langiappe:

- What are:
  - The collar's vertical displacement as a function of time?
  - The collar's total distance traveled?

Particle Kinetics: Cartesian Example ref ~Meriam & Kraige 3/5

Langiappe (continued):

- Returning to the expression for v_y=f(t)

$$v_{yf} = \frac{dy}{dt} = \frac{\mathrm{F}}{\mathrm{mk}}\{[1-\cos(kt)]\mu_k - \sin(kt)\} + gt$$

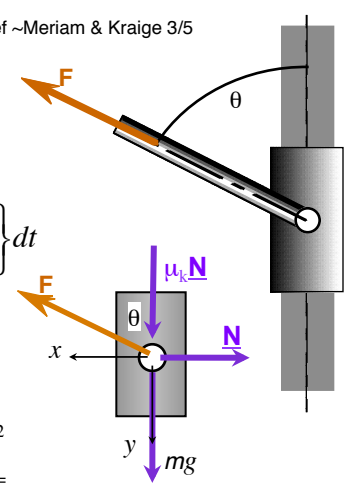
$$\int_0^{y_f} dy = \int_0^t \left\{\frac{\mathrm{F}}{\mathrm{mk}}\{[1-\cos(kt)]\mu_k - \sin(kt)\} + gt\right\} dt$$

$$y_f = \left[\frac{\mathrm{F}}{\mathrm{mk}}\{[t - sin(kt)]\mu_k + \cos(kt)\} + \frac{1}{2}gt^2\right]_0^t$$

$$y_f = \frac{\mathrm{F}}{\mathrm{mk}}\{[t - sin(kt)]\mu_k + \cos(kt) - 1\} + \frac{1}{2}gt^2$$

$$y_f = \frac{\mathrm{F}}{\mathrm{mk}}\{[\frac{\pi}{2k} - sin(\frac{\pi}{2})]\mu_k + \cos(\frac{\pi}{2}) - 1\} + \frac{1}{2}g\left(\frac{\pi}{2}\right)^2$$

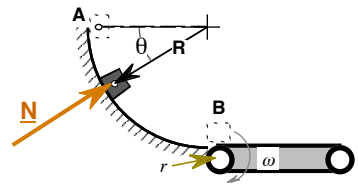
$$y_f = \frac{\mathrm{F}}{\mathrm{mk}}\{[\frac{\pi}{2k} - 1]\mu_k - 1\} + \frac{1}{2}g\left(\frac{\pi}{2}\right)^2$$



Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/7

Given:

- A box of mass m (particle) is released from rest @ top of a smooth circular track.



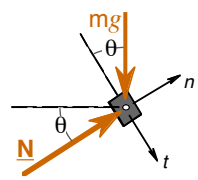
Find:

(A) Normal force **N** as a function of position θ along the circular track.

(B) The angular velocity (ω) of the pulley B such that the boxes don't slide onto the conveyor belt

Solution:

- FBD of box in general position - working over motion interval *A-B,* then instantaneous @ B!
  - Track smooth ==> μ=0, no friction!
- Attach normal-tangential coordinate axes



- Newton's Laws for general θ

$$\sum F_n = N - mg\sin\theta = ma_n$$

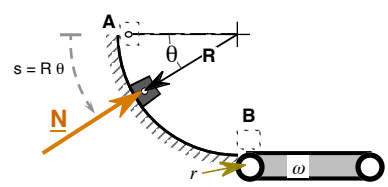
$$\sum F_t = mg\cos\theta = ma_t$$

- Must use kinematics to resolve unknowns!

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/7

Solution (cont'd):

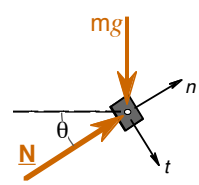
- Kinematics: Circular track-> path coord



$$\sum F_n = N - mg\sin\theta = ma_n = m\frac{v^2}{R}$$

$$\sum F_t = mg\cos\theta = ma_t = m\dot{v} = m\ddot{s}$$

- Using tangential direction to resolve velocity by integrating alternate form => $vdv = a_t ds$



$$a_t = \ddot{s} = g\cos\theta \quad s = R\theta \Rightarrow ds = Rd\theta$$

$$\int_0^v vdv = \int_0^\theta gR\cos\theta d\theta$$

$$\frac{1}{2}v^2 = gR\sin\theta\Big|_0^\theta = gR\sin\theta$$

$$v^2 = 2gR\sin\theta$$

- Now using the normal direction

$$\Rightarrow N = m\frac{v^2}{R} + mg\sin\theta$$

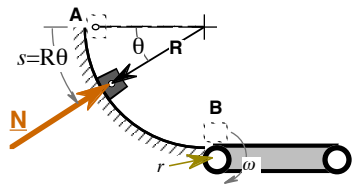
$$= 2mg\sin\theta + mg\sin\theta$$

$$\text{(A)}\ N = \underline{\underline{3mg\sin\theta}}$$

Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/7

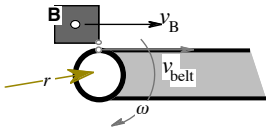
Solution (cont'd):

- Belt Kinematics:
  - From previous page, the box speed as it reaches B



$$v = \sqrt{2gR\sin\theta}\Big|_{\theta=\frac{\pi}{2}} = \sqrt{2gR}$$

  - Belt must move at the same speed to avoid slip, i.e. relative velocity between belt/box is zero



$$v_B = v_{belt}$$

$$\sqrt{2gR} = \omega\ r$$

$$\text{(B)}\quad \omega = \frac{\sqrt{2Rg}}{r}$$