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Secure Implementation in Production Economies

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Abstract

One thing that has been assumed for a long time is that whenever there is dominant strategy equilibrium in the game form of any mechanism and the outcome corresponding to that strategy profile is socially optimal, people will play that particular equilibrium strategy profile. The theory has been silent on why they will play that particular strategy profile when there are other (Nash) equilibria. The Nash/Bayes' Nash implementation being a possible solution to this problem suffers from the drawback of either the requirement of the designer knowing the (common) prior (in case of Bayes' Nash implementation) or the requirement of the players predicting the actions of other players and collaborate without pre-talk (in case of Nash implementation with absence of dominant strategy or unique Nash).

Secure implementation [Saijo et al. (2007)] is a relatively new concept in the theory of mechanism design and implementation. This requires double implementation in Dominant Strategy Equilibrium and Nash Equilibrium by the same Mechanism. This concept has worked well in some particular environments and has been tested on data [Cason et al. (2006)]. Unsurprisingly, being stronger than both the two above said concepts of implementation, there are many impossibility results in specific environments with richer domains. We look for secure implementability in production economies with divisible goods. We find that a very broad generalization of "Serial" Social Choice Function (SCF) [Moulin and Shenker (92)] as defined in [Shenker (92)] is securely implementable. We call such functions as Generalized Serial SCF (GSS). We also find that under certain conditions the Fixed Path SCFs are special cases of GSS and thus they are also securely Implementable. We conjecture that these are the

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only securely implementable SCFs in our environment if we add a few desirable axioms.

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JEL classification: C70, C72, D62, D71, D78, H42.

1 Introduction

We consider the standard implementation problem where an outcome has to be chosen from a set of alternatives depending upon the characteristics (e.g., preferences) of the agents in the society. The rule which chooses this outcome based on the true preference profile[1] (or any other such characteristic[2]) of the agents is called a Social Choice Function (SCF). The problem of implementing this rule arises because the above said "characteristics of the agents" may be private information of these agents and it may not be in their best interest to reveal these true characteristics if they know how the outcome is going to be chosen based on their reports. To achieve the goal of implementing a SCF it may be the case that the agents are directly asked to report their preferences or they may be asked to indulge in an indirect process where they interact under certain rules. In both the cases the institution which is used creates a game amongst the agents. These institutions are called mechanisms or game forms. The case where agents are required to report their preferences directly and the outcome is chosen according the SCF is called direct mechanism. The other one is called indirect mechanism. Strategyproofness of direct mechanisms is a requirement on the mechanism that truth telling by each agent leads to a most favorable outcome for that agent, no matter what the other agents are reporting. In other words, truth-telling is a dominant strategy equilibrium under the mechanism if the mechanism is strategyproof. It seems natural that players will reveal the truth if it is dominant strategy to do so. However, the performance of strategyproof mechanisms in achieving socially desired goals has been in question for a long time. On the one hand, a sequence of experiments show that pivotal mechanisms[3] fail to get the truth telling as a unique outcome (see Attiyeh et al. [2], Kawagoe and Mori [16], etc). On the other hand, there are experiments which show that true valuation is not revealed by the subjects in second price auction[4] experiments (see Kagel et al. [13], Kagel and Levin [14]). Some experimentalists argue that the subjects who don't play their dominant strategy must be confused by the complexity of the mechanisms where the dominant strategy may not be that clear. But neither epistemic(Aumann

[1] A preference profile is a set of preferences – one for each agent.

[2] In our framework, the characteristics of the agents we are considering are their preferences. But, more generally it can be the agents' endowments, the agents' abilities (e.g. production technology) etc.

[3] Pivotal mechanisms are strategyproof mechanisms in the problem of provision of public goods

[4] Second price auction is another example of strategyproof mechanism where the highest bidder gets the object and pays the highest losers bid. Others pay nothing.

and Brandenburger [1]) nor evolutive(Hurwicz [12], Smith [27]) models of game theory provide unambiguous support for the elimination of weakly dominated strategies. In fact the only prediction that is supported in these models is that the outcome must be a Nash Equilibrium (NE).

This leads us to think of two problems associated with strategyproof mechanisms. First, truth-telling may not be an agent's unique dominant strategy and using wrong dominant strategy may lead to wrong outcomes. Second, there can be NE other than the dominant strategy equilibrium which lead to wrong outcomes. To see this problem, consider a simple example of pivotal mechanism for two players. Suppose there is a costless public project to be undertaken if and only if the sum of the (reported) valuations of the project by the two agents is non negative. It is well known since Clarke [4] that the transfers according to pivotal mechanism[5] induce truth telling as a dominant strategy equilibrium. It is fairly easy to see that no one can gain by reporting anything other than the true value irrespective of what the other is reporting. However, the true profile is not the only NE. As a matter of fact, as is demonstrated in figure 1 below, almost half of the two dimensional Euclidian space constitute the set of NE. Here, the axes represent the type (valuation) space of the agents and since the pivotal mechanism is a direct mechanism they also represent the strategy space of the agents. Notice that the area which correspond to the set of NE (the shaded area) has two regions. In the first region (which is shaded green), the corresponding outcome is socially desired. However, there exists another region (which is shaded yellow) of the similar size where the NE leads to outcome which is not socially desired.

Secure implementation (Saijo et al. [25]) is one way to remedy this problem faced by the strategyproof mechanisms. Secure implementation of a SCF requires the existence of a mechanism under which there is a dominant strategy equilibrium which leads to the socially desired outcome and all the NE under the mechanism also lead to the socially desired outcome. This mode of implementation has been tested on data and has been found to be performing significantly better than strategyproof mechanisms under the presence of multiple NEs (Cason et al. [5]). This nice property of secure implementation doesn't come without costs. In many environments there doesn't exist non trivial SCFs which are securely implementable. Following are some of the examples.

Consider a public good provision problem where the good must be provided if and only if the sum of valuations is non-negative. We have just seen above that the pivotal mechanism doesn't securely implement this SCF. Notice that this SCF is efficient i.e. it maximizes the social surplus. It has been shown in theorem 7 in (Saijo et al. [25]) that there doesn't exist any surplus maximizing SCF which can be securely implementable[6]. This negative result of incompati-

[5] A transfer t_i to agent i according to pivotal mechanism in this environment will be equal to the $-|v_j|$ if agent i is pivotal i.e. absence of agent i would have changed the decision of undertaking the social project. Here, v_k is the valuation of agent k for the public project. In other words, If the presence of an agent alters the outcome in her favor, she must compensate the others for their (revealed) welfare loss.

[6] This result is valid even when the consider multivalued Social Choice Correspondences

bility between surplus-maximizing and secure implementation in the quasilinear environment[7] with discrete social decision is further illustrated by the second price auction where a large set of NE correspond to the non-surplus-maximizing outcome[8]. To see this point, consider a two player example where the valuation for the object to be auctioned are θ_1 and θ_2 by agent 1 and agent 2 respectively. Suppose, $\theta_1 > \theta_2 > 0$. In order to maximize the total surplus, it must be the case that object is allocated to agent 1. But, as we see in figure 2 below, the set of NE is quite large. The lower right set of NE correspond to the surplus maximizing outcome whereas the upper left set of NE end up allocating the object to agent 2.

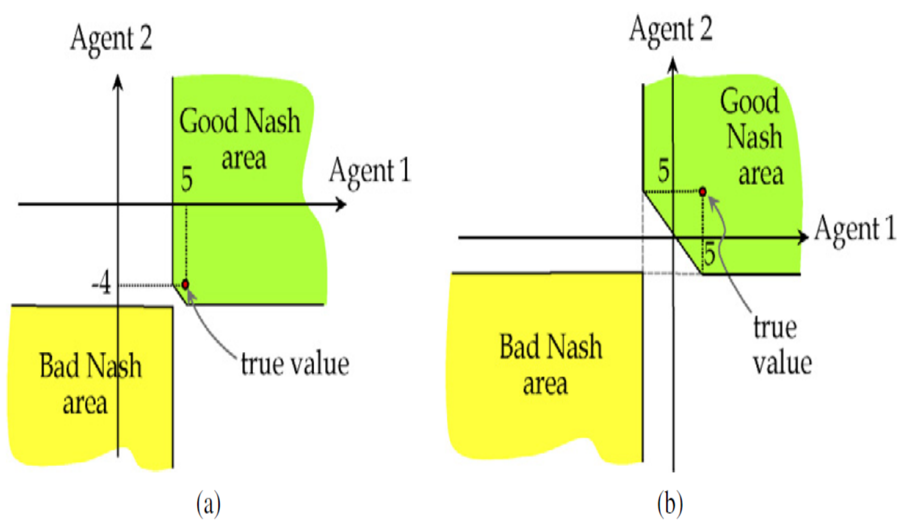

Figure 1: Equilibria of the pivotal mechanism

Another environment is where the social decision is a continuous variable but there are no transfers involved. Consider a *single-peaked voting environment* where the set of alternatives is $A = [0, 1]$ and set of possible preferences are those that are continuous and single-peaked[9] on A. In such an environment one can find nice SCFs which are strategyproof such as the *median voter rule*[10]. Median voter rule enjoys nice properties like pareto-efficiency, non-dictatorship, non-bossiness apart from strategyproofness (in fact, group-strategyproofness).

(SCC) in place of SCFs.

[7] Quasilinear preferences are represented by utility function which is additive and linear in one commodity called money.

[8] Surplus maximization here means that the private good to be auctioned must be allocated to the agent with the highest valuation.

[9] Single-peaked preferences requires the existence of a point $\mathrm{p}(\mathrm{u}_i)$ for each i called the peak of agent i with preference u_i such that u_i is strictly increasing before $\mathrm{p}(\mathrm{u}_i)$ and strictly decreasing after $\mathrm{p}(\mathrm{u}_i)$.

[10] Median voter rule picks the median of $\{\mathrm{p}(\mathrm{u}_i)\}_{i \in N}$ given a profile u.

However, this rule is not securely implementable. Similarly, other well known-known rules, such as the one which picks the smallest of the peaks, are not securely implementable. As a matter of fact, it has been shown in theorem 8 of (Saijo et al. [25]) that only rules which are securely implementable in this environment are the dictatorial rules. If we relax the rule and allow it be multivalued then we can get non-dictatorial rules but they can not be pareto-efficient.

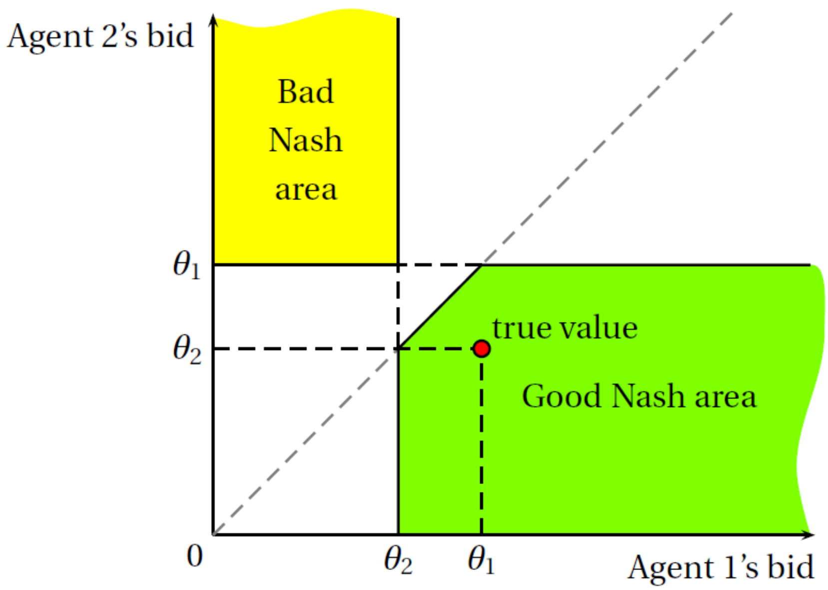

Figure 2: Equilibria of the second-price auction

There are more recent negative results. Bochet and Sakai [3] show that in allotment economies the securely implementable rules are either efficient (priority rules) or symmetric (equal division) but not both. They also show that in "uniform rule" bad Nash Equilibria can be avoided but for that we need to allow for pre play talk among the players which is a different set up of implementation in itself. Fujinaka and Wakayama [11] show that in an economy with indivisible objects and money, the only securely implementable rules are the constant rules.

The above examples show how difficult it is to find securely implementable rules which have other nice properties. However, there are environments where securely implementable rules do exist. For example, such environments are found in quasilinear setup where the social decision is a continuous variable. It has been shown in Saijo et al. [25] that serial SCF (Moulin and Shenker [22]) is securely implementable in the one input one output production economy with convex cost technology. We look for other possible SCFs which are securely implementable in such environments. We find out that it is not just the serial SCF which is securely implementable but a class of SCFs called generalized serial SCFs (GSS) defined in (Shenker [26]) are also securely implementable when the technology has convex cost.

The generalized serial SCFs are described for production economies with smooth production technologies. By smooth production technology we mean that the feasible consumption bundle must lie on some smooth manifold in R^m. Each agent reports his utility function which is defined over R^m_+ and is non-decreasing, continuous, locally non-satiated and quasi-concave. Then, the mechanism allocates the set of feasible bundles corresponding to the unique NE of an underlying game. The GSS are more general than other generalizations of Serial Mechanism whose incentive properties have been studied in the literature. Among these are the Fixed Path Methods (FPM) where the share of total cost paid by an agent is decided by a path in R^n_+ where n is the number of agents. This path does not depend on the demand vector. We find out that under some assumptions on the cost functions and on the preferences (which guarantee the desired incentive properties of such methods), the FPMs are in fact special cases of GSS and thus all such FPMs are securely implementable. However, there are GSS which can not be represented as Fixed Path Methods. We conjecture that if we require the mechanism to be non-constant, symmetric (anonymous) and smooth then GSS are the only mechanisms which are securely implementable.

At this point, it is very important to note the intuition why the serial SCF (or more generally the GSS) have such nice incentive property of secure implementability whereas, as we will discuss later, the SCF corresponding to other well known cost sharing rules like the Aumann-Shapley rule (which is the proportional rule in homogeneous goods case) does not share this feature[11]. In the latter, by changing the report an agent can affect the outcome for all the agents simultaneously. In particular, that agent's report changes the outcomes of such agents whose report in turn can change his outcome. This severe nature of externality in such SCF violates the acyclicity condition necessary for the combination of non-bossiness and strategyproofness (see Satterthwaite and Sonnenschein [29]) of the SCF which in turn is necessary for secure implementability. Under the serial SCF, on the contrary, the protection of lower demanders[12] from the demands of the higher demanders makes the externality one sided which is not that severe. More precisely, a change in the report of low demander changes the outcomes for all the high demanders whereas small change in the report of high demanders doesn't affect the outcome for the lower demanders.

The rest of the paper is arranged as follows. In section 2 we precisely introduce the notion of secure implementability and give one proposition which characterizes the securely implementable SCFs. In section 3 we define the serial cost sharing method and introduce some generalizations considered in literature with special focus on the ones whose strategic properties have been studied. In

[11] The SCF corresponding to the Aumann-Shapley rule is not even strategyproof

[12] By low demander in homogeneous case we mean an agent who gets smaller share of the output and pays lower level of input as the final outcome of the SCF. In heterogeneous case, all the generalizations of serial mechanism rank the agents in an order based on different criteria.

section 4 we define serial SCF and GSS and in section 5 we present two of our main results. The main proofs are gathered in the appendix A.

2 Secure implementability

We consider an arbitrary set of *alternatives* A and a finite set of *agents* $N = \{1, 2, ..., n\}$, where $n \geq 2$. Typical agents are represented by alphabets i, j etc. The preference relation of agent i over the set A is represented by *utility function* u_i. The set of *admissible utility functions* for agent i is denoted by U_i.The cartesian product of $U_1, U_2, ..., U_n$ is represented by U i.e. $U \equiv \underset{i \in N}{\times} U_i$. A typical element of U is a *utility profile* $u = (u_1, ...u_n)$ which is an n-tuple of utility functions– one for each agent. A *social choice function* (SCF) $f : U \to A$, is a function that associates with every $u \in U$ a unique alternative $f(u)$ in A. A *mechanism* (or a game form) $g : S \to A$ is a function that assigns to every $s \in S$ a unique element of A, where $S \equiv \underset{i \in N}{\times} S_i$ and S_i is the strategy space of agent i.

Definition 1 *The mechanism g is called a direct revelation mechanism associated with the SCF f if $S_i = U_i$ for all $i \in N$ and $g(u) = f(u)$ for all $u \in U$.*

Some times we may refer a direct revelation mechanism as the SCF if no confusion arises. When the strategies of agents $j \neq i$ is fixed at $s_{-i} \equiv (s_1, s_2, .., s_{i-1}, s_{i+1}, ..., s_n)$, agent i can induce certain outcomes by choosing strategies from the set S_i. The set of such outcomes denoted by $g(S_i, s_{-i})$ is called the attainable set or the *opportunity set* of agent i at s_{-i}. More formally, $g(S_i, s_{-i}) \equiv \{b \in A | \exists s_i \in S_i \text{ s.t.} g(s_i, s_{-i}) = b\}$. The set of alternatives that agent i with utility u_i ranks weakly below the alternative $a \in A$ is called the weak lower contour set for agent i with utility u_i at a and is denoted by $L(a, u_i)$. More formally, $L(a, u_i) \equiv \{b \in A | u_i(a) \geq u_i(b)\}$. Given the mechanism $g : S \to A$, the strategy profile $s \in S$ is *a Nash Equilibrium (NE)* of g at $u \in U$ if $\forall i \in N$, $g(S_i, s_{-i}) \subseteq L(g(s), u_i)$. Let's denote by $N^g(u)$ the set of Nash equilibria of g at u.

Definition 2 *The mechanism g implements f in Nash equilibria if for all $u \in U$, (i) $\exists$ $s \in N^g(u)$ st. $g(s) = f(u)$ and (ii) $\forall s \in N^g(u)$, $g(s) = f(u)$.*

The SCF f is *Nash implementable* if there exists a mechanism that implements f in Nash equilibria. Given the mechanism $g : S \to A$, the strategy profile $s \in S$ is *a Dominant strategy Equilibrium of g at $u \in U$* if $\forall i \in N$,$\forall \tilde{s}_{-i} \in S_{-i}$, $g(S_i, \tilde{s}_{-i}) \subseteq L(g(s_i, \tilde{s}_{-i}), u_i)$. Let's denote by $DS^g(u)$ the set of dominant strategy equilibria of g at u.

Definition 3 *The mechanism g implements f in Dominant Strategy equilibria if for all $u \in U$, (i) $\exists$ $s \in DS^g(u)$ st. $g(s) = f(u)$ and (ii) $\forall s \in DS^g(u)$, $g(s) = f(u)$.*

The SCF f is *Dominant Strategy implementable* if there exists a mechanism that implements f in Dominant Strategy equilibria. We now introduce formally the concept of secure implementation which requires the existence of a mechanism which implements the the SCF in Nash equilibria as well as in Dominant Strategy equilibria.

Definition 4 *The mechanism g securely implements the SCF f if for all $u \in U$, (i) $\exists\ s \in DS^g(u)$ st. $g(s) = f(u)$ and (ii) $\forall s \in N^g(u)$, $g(s) = f(u)$.*

The SCF f is *securely implementable* (SI) if there exists a mechanism that securely implements f. Strategyproofness is a requirement on a SCF that truth telling by the agents is a dominant strategy under the direct revelation mechanism. More formally, the SCF f satisfies *strategy proofness* (SP) if, $\forall u \in \mathbf{U}, \forall i \in N, \forall \widetilde{u}_i \in \mathbf{U}_i, u_i(f(u)) \geqslant u_i(f(\tilde{u}_i, u_{-i}))$. Another technical property on the SCF, introduced in Saijo e.t. al. [25], which together with strategyproofness characterizes secure implementability is called rectangularity property and is defined as following. The SCF f satisfies the *rectangularity property* (RP) if for all $u, \tilde{u} \in U$, if $u_i(f(\tilde{u}_i, \tilde{u}_{-i})) = u_i(f(u_i, \tilde{u}_{-i}))$ for all $i \in N$,then $f(\tilde{u}) = f(u)$. The following characterization due to Saijo e.t. al. [25] will be used in one of our main results.

Proposition 1 (Saijo e.t. al. [25]): *A SCF f is Securely Implementable if and only if f satisfies Strategyproofness and Rectangularity Property.*

3 Serial Cost Sharing Methods

Serial cost sharing method was first introduced for an environment where the goods demanded by the agents are homogeneous or, in other words, the agents demand various quantities of the same good. Since our purpose here is to extend this method to more general settings, we will define the problem in an environment where each agent $i \in N$ demands $q_i \in [0, q^{\max}] \subset R_+$ quantity[13] a personalized[14] good i. Thus q_i, the ith component of vector $q \in R_+^N$,can be thought of as the demand for good i as well as the demand of agent i. The cost of serving these demands is $C(q)$, which must be divided among the agents; the cost share of agent i is given by $x_i(q; C)$. The preferences of agent i is defined over R^2 which is continuous, increasing in q_i, decreasing in x_i and the upper contour set is convex[15]. Let a concave utility function $u_i(q_i, x_i)$ represent the preference of agent i. Recall that for the homogeneous goods case $C(q) = C(q_N)$ where, $q_N = \sum_{i \in N} q_i$. Here the serial cost sharing method is defined as follows. Consider, Without loss of generality $q_1 \leq q_2 \leq .... \leq q_n$. Define, $q^i = (q_1, q_2, ..., q_{i-1}, q_i, q_i, ..., q_i)$ then ,

[13] $q^{\max}$ can be ∞

[14] In some of the more general models e.g., [17], [15], each agent may demand quantities of some or all of the goods.

[15] An special case which is widely studied in this framework is the preference which is quasilinear in x_i and concave in q_i.

$$x_i(q;C) = \frac{C(q^i)}{n+1-i} - \sum_{k=1}^{i-1} \frac{C(q^k)}{(n+1-k)(n-k)} \tag{1}$$

This method works as follows. Agent 1, with the lowest demand q_1 pays $1/n$ th of the cost of nq_1. Agent 2, with the second lowest demand pays agent 1's cost share, plus $1/(n-1)$th of the incremental cost from nq_1 to $q_1 + (n-1)q_2$. Agent 3, with the next lowest demand pays agent 2's cost share, plus $1/(n-2)$ th of the incremental cost from $q_1 + (n-1)q_2$ to $q_1 + q_2 + (n-2)q_3$. And so on. This method is characterized by "anonymity" and "invariance of the cost share of low demanders by a change in the demand of high demanders". The demand game generated by this method is as follows. Each agent has a strategy (demand) space which is R_+ and his cost share as a function of the demand profile is computed by (1). The payoff is given by the utility function defined above. It should be noted that the serial cost sharing method (1) is defined for any arbitrary cost function. However, if we assume the cost function to be strictly[16] convex (increasing marginal costs), then this demand game has very strong strategic properties. In this demand game the NE is unique, robust to coalitional deviations and the only rationalizable strategy profile. Moreover, this NE is the unique outcome of adaptive learning (Milgrom and Roberts, [19]).

Given the nice strategic and equity properties that the serial method enjoys in homogeneous good setting, it is natural to look for the extension for the rule in more general settings. In particular, a natural question is what the counterpart of the serial method in heterogeneous good (multidimensional) case will be. Among the various approaches to extend the Serial Mechanism to the case of heterogeneous goods, it is a general consensus (Koplin [15], Koster et al. [17], Sprumont [28], Friedman [7], Friedman and Moulin [9] etc.) that the mechanism must coincide with the Serial Mechanism in the homogeneous case. This property is referred to as *serial extension.* But, the task of extending the serial mechanism to heterogenous goods case is not an easy one as was first demonstrated by Koplin [15]. He shows using a nice counterexample that *serial extension* is not compatible with other desired properties namely *consistency (*he calls it *direct aggregation invariance), scale invariance* and *additivity* each of which is compelling in its own sense. Consistency is the requirement from the cost sharing method that the cost shares be invariant if we relabel the commodities. Scale invariance requires that the units in which the goods are measured does not affect the cost shares. Additivity is a decentralizability axiom which says that if we can separate the production into different processes, then we should be able to apply the same cost sharing method in each process and the still get the same cost shares. Therefore, knowing that we can not be too demanding with respect to serial extension there have been different approaches to pin down the class of methods which carry on the properties of serial method

[16] strictness is not needed if the preferences of the agents are strictly convex.

for homogeneous methods to the heterogeneous goods environment. These approaches can be broadly categorized into two groups– one which focusses on axiomatic approach (Koster et al. [17], Sprumont [28]) and the other which is concerned about the strategic properties (Friedman [7], Friedman and Moulin [9], Friedman [8]).

Since we are more interested in the strategic properties, we analyze the second approach. Friedman [7] studies the strategic properties of these methods which we describe in the next paragraph and finds out that these do enjoy nice strategic properties similar to serial cost sharing in homogeneous goods case. He finds out that the game induced by such methods is solvable by iterative elimination of *overwhelmed strategies*[17] introduced in Friedman & Shenker [10][18].

This natural extension of the serial method (1) to the heterogeneous case, where $C(q)$ is an arbitrary non-decreasing and continuously differentiable function of its n variables, which was introduced in Friedman and Moulin [9], is defined as follows. Consider a path[19] γ^{SC} from 0 to q given by $\gamma^{SC}(t;q) = (te) \wedge q$, for $t \geq 0$, where $(p \wedge q)_i = \min\{p_i, q_i\}$ and $e = (1, 1, ..., 1)$ is the unit vector in R^N. This path essentially follows the diagonal of the n-dimensional positive orthant till its coordinates are smaller than all the coordinates of the demand vector q. As soon as it meets the demand of some agent, it starts following the projection of the diagonal in the hyperplane where that coordinate is fixed at the demand in that coordinate and so on. Given such a path γ^{SC} the cost sharing mechanism is given by,

$$x_i^{SC}(q;C) = \int_0^\infty \partial_i C(\gamma^{SC}(t;q)) d\gamma_i^{SC}(t;q) \tag{2}$$

Here, $\partial_i C(p)$ is the partial derivative of C with respect to p_i evaluated at p. It is clear from (2) that the path relevant to an agent is independent of higher demands. Thus, the cost shares of agents are unaffected by small changes in the demands of higher demanding agents. Therefore, the externality is one sided (and thus, acyclic). Intuitively, due to this reason this mechanism enjoys nice strategic properties that we will see in Theorem 1. Moreover, due the same reason, the nice strategic properties are preserved if the γ^{SC} is replaced by any arbitrary continuous non-decreasing path $\phi(t;C) \wedge q$, where ϕ satisfies the following properties. For fixed C, ϕ is non-decreasing and continuous in t with $\phi(0;C) = 0$ and $\lim_{t\to\infty} \phi_i(t;C) > q^{\max}$ for all i. See figure 3 below for an example of such a ϕ.

[17] A strategy s_i for agent i is overwhelmed by strategy $\tilde{s}_i$ with respect to S_{-i} if the best that agent i can get over S_{-i} by playing s_i is worse than the worst that he gets by playing $\tilde{s}_i$.

[18] Notice that this is stronger property than solvability in elimination of dominated strategies.

[19] SC in the symbol underlines the point that this path corresponds to the generalization of Serial Cost (SC) sharing rule.

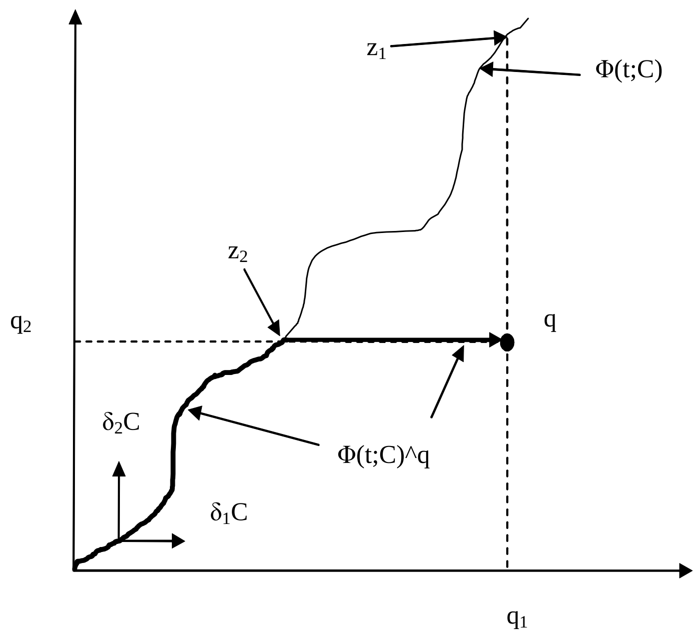

Figure 3: Fixed path method in two agent case

This liberty of choosing the ϕ gives rise to a huge class of cost sharing methods called Fixed Path Methods (FPM). There is a FPM corresponding to each fixed path ϕ which can be defined as follows

$$x_i^{\phi}(q;C) = \int_0^{\infty} \partial_i C(\phi(t;C) \wedge q) d(\phi_i(t;C) \wedge q_i) \tag{3}$$

These are called fixed path methods because the path ϕ, which does not depend on q and thus, are in a sense fixed, uniquely defines a method. One example of a fixed path is the path which follows the the edges of the rectangle $[0.q]$ in some predecided order and this leads to the incremental methods. Notice that when cost function is symmetric or when ϕ is independent of the cost function then the only symmetric FPM is the Friedman-Moulin method (2) defined by the path which is the diagonal of the positive orthant. Leroux [18] provides a justification of non-symmetric paths. However, symmetry is trivially satisfied when the cost function is not symmetric and we allow ϕ to be a function of C. This gives rise to a huge class of symmetric methods. Clearly, we will be sacrificing *additivity* in most of the cases but we can recover scale invariance and even stronger properties like *ordinality*[20] (see Sprumont [28]). The path which most closely follows the spirit of serial method is the path which defines Moulin-Shenker ordinal method discussed in (Sprumont [28]). This path which we will call ϕ^{MS} is defined by the solution of the following differential equation

$$d\phi_i^{MS}(t;C)/dt = 1/\partial_i C(\phi^{MS}(t;C))$$

satisfying the boundary condition $\phi^{MS}(0;C) = 0$. This path has the property that on any point on the path the incremental cost generated by a small move along the path is shared equally among the agents not fully served. Other examples of FPMs can be generated by applying a FPM to any suitably normalized problem e.g. applying FPM to axially normalized problem (Friedman [7]). One seemingly natural FPM thus generated discussed in Friedman [7] is the use of diagonal path after axial normalization of the problem.

Given the set of agents N, utility profile $u = \{u_i\}_{i\in N}$, a cost function C, a fixed path method $x^{\phi}(\cdot;\cdot)$ induces a cost sharing game $\Gamma(x,u)$. These induced games have variety of strategic properties: uniqueness of NE, Strong Equilibria, uniqueness of set of rationalizable outcomes and convergence of adaptive learners. Friedman [7] shows these properties for fixed path methods by showing that the induced games are O-Solvable which in turn implies all these properties.

Theorem 1 (Friedman [7]): *Assume that the marginal cost ($\partial_i C(q)$) is strictly increasing in all variables, $x_i^{\phi}(\cdot;\cdot)$ is a fixed path method and that preferences, $u_i(q_i,x_i)$ are increasing in q_i, decreasing in x_i, and concave. Then the induced game is O-solvable.*

It should be noted that there can be paths which depends on q and we can use such paths to define "path methods" in a similar fashion as (3). One prominent example of such a path method is the Auman-Shapley method where the path is the ray joining the origin to the demand vector, thus for each demand there corresponds a path. More precisely, the path which generates the Aumann-Shapley method is given by $\phi^{AS}(t;C)(q) = tq$. We notice that this path is not a fixed path and the demand game generated by this method does not share the appealing strategic properties that is enjoyed by the FPMs. The Aumann-Shapley method in the homogeneous goods case is the proportional method. It has been shown in Watts [30] (see also Moulin [20] for detailed analyses) that uniqueness of NE is not guaranteed in the proportional demand games for general convex preferences and a sufficient condition has been shown to be the binormality of preferences. Moreover, as we will discuss in next section, even when the NE is unique this method doesn't share the strategic properties of that of the FPMs. Intuitively, this happens because a change in the demand by any agent changes the cost shares of all the agents. For more on such path

[20] Ordinality is a stronger requirement than scale invariance. Scale invariance requires that the cost shares should be invariant to linear transformation of the demand profile whereas ordinality requires that it should be invariant to any arbitrary monotonic transformations, possibily non linear.

methods and axiomatic characterization of methods generated by paths and more generally by convex combinations of paths please refer to Friedman and Moulin [9].

4 Serial SCF and generalized serial SCF

We mentioned in the last section that if the production technology has increasing marginal costs and the preferences are convex then the serial rule (1) defined in the homogeneous goods case induces a game which admits a unique NE. A serial social choice function (SCF) for a fixed cost function C associates this unique NE allocation to the preference profile generating this game.

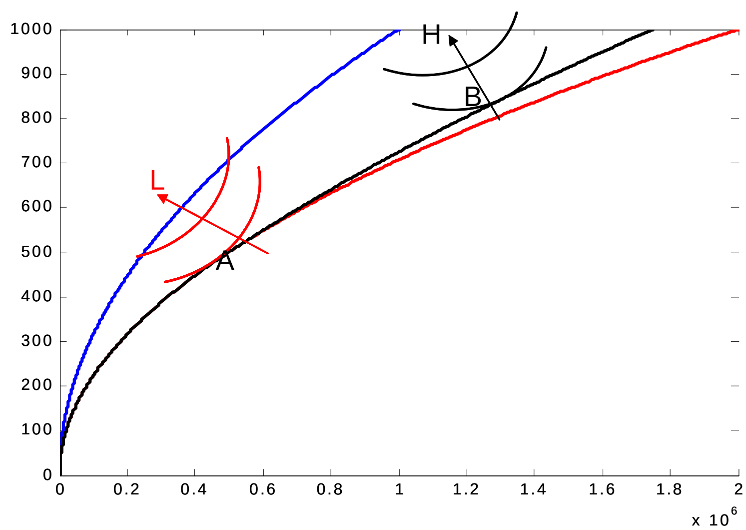

Figure 4: Serial SCF in two agent case

Figure 4 above demonstrates the Serial mechanism (SCF) in two individual and two good economy where one good x is the input (x-axis) and the other good q is output (y-axis). The production technology is decreasing returns to scale i.e. the cost function is convex. The blue curve is c(q), red one is c(2q)/2 and the black one coincides with the red one till point A and then goes parallel to blue curve. More precisely, the black curve has two parts. The part below A is the locus of points that are 1/2 of some point on the blue curve. The part above A is the locus of points whose vector sum to the point A belongs to the blue curve. The high valuation agent (H) is the agent whose MRS is higher for the output with respect to the input. The other agent is the low valuation agent (L). The agents are required to report their utility functions and allocation is assigned according to the Serial cost sharing rule. More details on the algorithm to implement the serial SCF can be found in Moulin and Shenker [22]. The purpose of bringing the 2X2 case of Serial SCF here is that the Generalized

Serial SCF is defined very closely in the spirit of Serial mechanism here. The three conditions below in the definition of generalized serial functions are linked to the following three observation in the above picture.

1) The opportunity set of L remains unaffected by change in the preferences of H as long as H has higher valuation than L.

2) Owing to the convexity of production function and the preferences, there is a unique maximizer point A for L on his opportunity set given H and also B for H on her opportunity set given L.

3) Owing to no kinks in red and black curve at A, A remains the optimum point for L even after small changes in preference by H.

Generalized Serial Mechanism (SCF):

The generalized serial SCF is defined for an economy with n agents and m goods where n and m are greater than 1. Production technology $\mathbf{P}$ is a $m-p$ dimensional smooth manifold which represents a technology where out of m-goods, p are inputs and m-p are outputs. Set of alternatives A is the set of allocation to the agents in N which is feasible under $\mathbf{P}$. More formally, $A \equiv \{x \in R_+^{mn} | (\sum_{i=1}^{n} x_i) \in \mathbf{P}\}$. One example of such set of alternatives where $m = 2$ and $p = 1$ is the set of allocations for the two agents in the above example which add up the a point which lie on the blue curve in figure 4. The set of admissible utility functions U_i for agent i contains the functions $u_i : R^m \to R$ that are continuous, non-decreasing in all dimensions, locally non satiated and quasi-concave. Linear utilities of agent i is the subset $\mathbf{L}$ of U_i which isomorphic

to $R_+^m - \{0^m\}$.

Definition 5 (Generalized Serial Function (Shenker) [26]) *Consider a function $a : R_+^n \to A$ st., $\forall z \in R_+^n$ and $\forall \lambda \in \mathbf{L}$:*

(1) $z_i \leq z_j \Rightarrow a_i(z) = a_i(z_{-j}, s_j), \forall s_j \in [z_i, \infty]$,

(2)$\lambda.a_i(z_{-i}, s_i)$ has a unique maximizer $\overline{s_i}$, $\forall i$

(3)If $\overline{s_i}$ is the unique maximizer of $\lambda.a_i(z_{-i}, s_i)$ then $\overline{s_i}$ is also the unique maximizer of $\lambda.a_i(z'_{-i}, s_i)$ $\forall z'$ st., $\forall j \neq i$, $MIN[z'_j, z_j] < \overline{s_i} \Rightarrow z'_j = z_j$.

Such a function "a" is called a "generalized serial function"

Let's denote by $\mathbf{F}$ the set of all generalized serial functions. For a given utility profile u a function $a \in \mathbf{F}$ induces the normal form game $\Gamma(a; u) \equiv \langle N,\ \forall i;\ S_i = R_+, \{u_i(a_i(\cdot)\}_{i \in N})\rangle$ where N is the set of players, R_+ is the strategy space for all players and the payoff function for player i is given by $u_i \circ a_i(\cdot)$. Such games possesses unique NE.

Lemma1 : *$\forall u \in U, \forall a \in F; \Gamma(a; u)$ has a unique NE.*

Proof: The proof consists of two steps. In first step it is shown that there can not be more than one NE and then an explicit algorithm is given to construct a NE. A formal proof can be found in Appendix A.1 below.

Definition 6 (Generalized Serial Mechanism) *ζ^a is a generalized serial mechanism (GSS) associated with $a \in \mathbf{F}$ if $\zeta^a(u) = a(z)$ where z is the unique NE of $\Gamma(a; u)$.*

Let's denote by **G** the set of all generalized serial mechanisms.

5 Secure implementability of Generalized Serial Mechanisms and the fixed path mechanisms

Now we are ready to present our main result which encompasses the result of Saijo et al. [25]

Theorem 2 : *Any generalized serial mechanism (GSS) is securely implementable.*

Proof: We show the secure implementabilty of GSS by showing that the GSS are strategyproof and that they satisfy the rectangularity property. Then by proposition 1 the desired result follows. Please refer to the Appendix A.2 below for a complete proof.

Now we define a class of social choice functions called fixed path social choice functions based on fixed path cost sharing rules. Lets assume the conditions on the cost function and the preferences that were used in theorem 1. Then from the theorem 1 we know that there will be unique NE in the game $\Gamma(x^{\phi}, u)$ induced by the the cost sharing rule x^{ϕ} based on the fixed path ϕ.

Definition 7 *A fixed path social choice function $\xi^{x^{\phi}}$ associates the allocation corresponding to the unique NE of the game $\Gamma(x^{\phi}, u)$ to the preference profile u.*

The following theorem states that all such fixed path SCF are securely implementable.

Theorem 3: *Under the assumptions of theorem 1, a fixed path social choice function $\xi^{x^{\phi}}$ is a special case of generalized serial social choice function and thus are securely implementable.*

Proof: The proof consists of explicitly constructing a generalized serial function "a" for every fixed path social choice function $\xi^{x^{\phi}}$. We use two lemmas for proving the desired properties of such "a". Please refer to Appendix A.3 below for a comprehensive proof.

At this time we would like to emphasize that the SCFs corresponding to path methods other than fixed path methods may not be securely implementable. One such method as we discussed above is the Aumann-Shapley method which corresponds to the proportional method in the homogeneous goods case. To ensure the uniqueness of NE in the demand game lets consider linear utilities (which are obviously binormal) given by $u_i(q_i, x_i) = b_i q_i - x_i$ and convex cost technology given by $c(y) = y^2/2$. Proportional cost shares are given by $x_i^{pr}(q) = \frac{q_i}{q_N} c(q_N)$, where $q_N = \sum_{i \in N} q_i$.Let's define the proportional SCF $\xi^{x^{pr}}$ which associates to every utility profile u the unique NE of the demand game $\Gamma(x^{pr}, u)$. We notice that this SCF is not securely implementable. As a matter of fact these are not even strategyproof. To see this consider a two agent situation. Let the linear utilities of agents 1 and 2 be defined by the parameters b_1 and

b_2. Then, whenever b_i's are close enough to ensure the active participation of both agents, the unique NE demand profile (q_1^*, q_2^*) is given by $q_i^* = \frac{4}{3}(b_i - \frac{b_j}{2})$ and the equilibrium cost shares turn out to be $x_i = \frac{4}{9}(b_i - \frac{b_j}{2}) + \frac{2}{3}b_j(b_i - \frac{b_j}{2})$, $i, j \in \{1, 2\}$. Therefore, the optimal report $\bar{b}_i^*$ of agent i with true parameter b_i is given by $\bar{b}_i^* = \frac{3}{2}b_i + \frac{5}{4}\bar{b}_j$ where $\bar{b}_j$ is the report of agent j. Clearly, there are profitable manipulation of reports by agents. In particular, suppose $b_1 = b_2 = b$ and agent 1 reports truthfully then the optimal report of agent 2 is $\frac{11}{4}b$.

We see that the FPMs are special case of GSS. However, there are GSS which can not be represented by FPM. One trivial example is a constant SCF. Therefore we conclude that GSS are more general than FPMs and have nice strategic properties..

We conclude by the following conjecture which we leave for future work.

Conjecture: *Every smooth, nonconstant, anonymous and securely implementable scf is an element of G.*

A Proofs

A.1 Proof of lemma 1

Step 1- Given any $u \in \mathbf{U}$ and any $a \in \mathbf{F}$; $\Gamma(a; u)$ can not have more than one NE.

Proof: Let z and $z\prime$ be two distinct NE with $z = (z_1, z_2, z_3, ..., z_n)$ and $z\prime = (z\prime_1, z_2', z_3', ..., z_n')$.

There must exist an element i such that $z_i \neq z_i'$ *and* $\min\{z_j, z_j'\} < \min\{z_i, z_i'\} \implies z_j = z_j'$.

Without loss of generality, say $z_i' < z_i$. But then $z_i' = \underset{s \in [0,1]}{argmax}\, u_i(a_i(s, z_{-i})) = z_i$ which is a contradiction.

Step 2- Given any $u \in \mathbf{U}$ and any $a(z) \in \mathbf{F}$; The following algorithm generates a profile z which is a (the) NE of $\Gamma(a; u)$.

Algorithm:

1)Set z=(1,1,....,1).

2) Define $s_i^1 = \underset{s \in [0,1]}{argmax}\ u_i(a_i(s, z_{-i})), \forall i$.

3) Without loss of generality, let $s_1^1 = \underset{i}{min}\{s_i^1\}$. Set $z_1 = s_1^1$ and leave the other elements of z unchanged.

4) Define $s_i^2 = \underset{s \in [0,1]}{argmax}\ u_i(a_i(s, z_{-i})), \forall i$.

5)Without loss of generality, let $s_2^2 = \underset{i \neq 1}{min}\{s_i^1\}$. Set $z_2 = s_2^2$ and leave the other elements of z unchanged.

6)Repeat the process to update $z_3, z_4, ..., z_n$.

Claim: The profile z obtained by the above algorithm is a NE of $\Gamma(a;u)$.
Proof:
Claim 1.If $s_i^i \leq s_{i+1}^{i+1}$ for all i = 1,2,...n-1, then z is a NE.
Proof: Straightforward from property 3 and the way z has been constructed.

Claim2. $s_i^i \leq s_{i+1}^{i+1}$ for all i = 1,2,...n-1
Proof:

Part1-$s_1^1 \leq s_2^2$. This holds because, $s_1^2 = s_1^1 = z_1$ (because 1 is solving the same optimization exercise) and $s_2^2 < s_1^1 \implies s_2^2 = s_2^1$ which contradicts the definition of s_1^1.

Part2- If $s_i^i \leq s_{i+1}^{i+1}$ for all $i < k$ then $s_k^k \leq s_{k+1}^{k+1}$.

Proof: Notice first that $s_l^k = z_l = s_l^l$ for all $l < k$. This is true because of condition 3. Now $s_{k+1}^{k+1} < s_k^k \implies s_{k+1}^{k+1} = s_{k+1}^k$ which contradicts the definition of s_k^k . ■

A.2 Proof of theorem 2

Strategyproofness of GSS follows from Theorem 7.2.3 in Dasgupta et al. [6], given our domain of preferences being monotonically closed and the fact that GSS is a single valued Nash Implementable SCF.

We will prove the Rectangularity Property:
$\forall u, \tilde{u} \in \mathbf{U}$; $\{u_i(\zeta^a(\tilde{u})) = u_i(\zeta^a(u_i, \tilde{u}_{-i})) \ \ \forall i \in N \implies \zeta^a(\tilde{u}) = \zeta^a(u)\}$

Proof:
Fix an arbitrary pair of utility profiles $u, \tilde{u} \in \mathbf{U}$

Let $u_i(\zeta^a(\tilde{u})) = u_i(\zeta^a(u_i, \tilde{u}_{-i})) \ \ \forall i \in N$

Define, $NE(\Gamma(a; u_i, \tilde{u}_{-i})) = \tilde{z}^i$; $NE(\Gamma(a;\tilde{u})) = \tilde{z}$; $NE(\Gamma(a;u)) = z$. (Notice the notation; $\tilde{z}^i$ is a vector and $\tilde{z}_i$ is the i'th component of the vector $\tilde{z}$. For example, $\tilde{z}_k^i$is the k'th component of $\tilde{z}^i$.)

Step1: $\tilde{z}^i = \tilde{z}$, $\forall i \in N$.

Proof: Let $\tilde{z}^i \neq \tilde{z}$ for some i.

Now, we must have an element k st. $\tilde{z}_k^i \neq \tilde{z}_k and \ \min\{\tilde{z}_j^i, \tilde{z}_j\} < \min\{\tilde{z}_k^i, \tilde{z}_k\} \implies \tilde{z}_j^i = \tilde{z}_j$.

Case1: $k \neq i$

Without loss of generality, say, $\tilde{z}_k^i < \tilde{z}_k$.

$\tilde{z}_k^i = \underset{s\in[0,1]}{argmax} \ \tilde{u}_k(a_k(s, \tilde{z}_{-k}^i)) = \underset{s\in[0,1]}{argmax} \ \tilde{u}_k(a_k(s, \tilde{z}_{-k})) = \tilde{z}_k$which is a contradiction.

Case2: $k = i$

Here there are two relevant cases,

Case2.1: $\tilde{z}_i^i < \tilde{z}_i$

Then we have,

$\tilde{z}_i^i = \underset{s\in[0,1]}{argmax}\ u_i(a_i(s,\tilde{z}_{-i}^i)) = \underset{s\in[0,1]}{argmax}\ u_i(a_i(s,\tilde{z}_{-i}))$

From property 1 in the definition of a, we must have the following

$a_i(\tilde{z}_i^i, \tilde{z}_{-i}^i) = a_i(\tilde{z}_i^i, \tilde{z}_{-i})$

or, $a_i(\tilde{z}^i) = a_i(\tilde{z}_i^i, \tilde{z}_{-i})$

$\implies u_i(a_i(\tilde{z}^i)) = u_i(a_i(\tilde{z}_i^i, \tilde{z}_{-i}))$

We also know, $u_i(\zeta^a(\tilde{u})) = u_i(\zeta^a(u_i, \tilde{u}_{-i}))\ \ \forall i \in N$

or, $u_i(a(\tilde{z})) = u_i(a(\tilde{z}^i))$, $\forall i \in N$.
or, $u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}^i))$, $\forall i \in N$.
Therefore, $u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}_i^i, \tilde{z}_{-i}))$
In other words, $u_i(a_i(\tilde{z}_i, \tilde{z}_{-i})) = u_i(a_i(\tilde{z}_i^i, \tilde{z}_{-i}))$
But then, $\tilde{z}_i^i = \tilde{z}_i$ because $\tilde{z}_i^i$is unique maximizer of $u_i(a_i(s,\tilde{z}_{-i}^i))$ and $u_i(a_i(s,\tilde{z}_{-i}))$.

Case2.2: $\tilde{z}_i^i > \tilde{z}_i$
From property 1 in the definition of a, we must have the following
$a_i(\tilde{z}_i, \tilde{z}_{-i}) = a_i(\tilde{z}_i, \tilde{z}_{-i}^i)$
or, $a_i(\tilde{z}) = a_i(\tilde{z}_i, \tilde{z}_{-i}^i)$
$\implies u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}_i, \tilde{z}_{-i}^i))$
We also know, $u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}^i))$, $\forall i \in N$.

Therefore we have,
$u_i(a_i(\tilde{z}^i)) = u_i(a_i(\tilde{z}_i, \tilde{z}_{-i}^i))$
But then, $\tilde{z}_i^i = \tilde{z}_i$ because $\tilde{z}_i^i$is unique maximizer of $u_i(a_i(s,\tilde{z}_{-i}^i))$ and $u_i(a_i(s,\tilde{z}_{-i}))$.

$\square$
Notice, the above step establishes the following property :
$\tilde{u}_i(a_i(s,\tilde{z}_{-i}))$ and $u_i(a_i(s,\tilde{z}_{-i}))$ both are maximized at $\tilde{z}_i = \tilde{z}_i^i$for all i —♠

Step 2:
$\zeta^a(\tilde{u}) = \zeta^a(u)$
Proof:
Proving $a(z) = a(\tilde{z})$ should be enough since, by definition $\zeta^a(\tilde{u}) = \zeta^a(u) \iff a(z) = a(\tilde{z})$.
In fact, we will prove a stronger property, namely, $z = \tilde{z}$.
Suppose not and let $z \neq \tilde{z}$.
Now, we must have an element k st. $z_k \neq \tilde{z}_k and\ \min\{z_j, \tilde{z}_j\} < \min\{z_k, \tilde{z}_k\} \implies z_j = \tilde{z}_j$.

There can be two cases,

Case 1. $z_k > \tilde{z}_k$

Then we get the following expression, where first and fourth equalities are from definition, second follows from the ♠ and third is due to the property 3 in the definition of function "a"

$\tilde{z}_k = \underset{s\in[0,1]}{argmax}\, \tilde{u}_k(a_k(s,\tilde{z}_{-k})) = \underset{s\in[0,1]}{argmax}\, u_k(a_k(s,\tilde{z}_{-k})) = \underset{s\in[0,1]}{argmax}\, u_k(a_k(s,z_{-k})) =$ z_k and we reach a contradiction.

Case 2. $z_k < \tilde{z}_k$

Then we get the following expression, where first and fourth equalities are from definition, third follows from the ♠ and second is due to the property 3 in the definition of function "a"

$z_k = \underset{s\in[0,1]}{argmax}\, u_k(a_k(s,z_{-k})) = \underset{s\in[0,1]}{argmax}\, u_k(a_k(s,\tilde{z}_{-k})) = \underset{s\in[0,1]}{argmax}\, \tilde{u}_k(a_k(s,\tilde{z}_{-k})) =$ $\tilde{z}_k$ and we hit another contradiction to conclude the proof.

■

A.3 Proof of theorem 3

We first present two lemmas which will be the key to the proof of theorem 3 below.

Lemma 2 (Lemma 1 in Friedman [7]): *Assume that marginal cost is strictly increasing in all variables and that $x_i^{\phi}(\cdot;\cdot)$ is a fixed path method. Define $z_i(q_i) = \min[t|\phi_i(t) \geq q_i]$. Then:*

(a) $x_i^{\phi}(q;C)$ is strictly increasing and strictly convex in q_i.

(b) $x_i^{\phi}(q;C)$ is non decreasing in q_j for all $j \neq i$.

(c) For all q and $\grave{q}_j$ such that both $z_j(q_j)$ and $z_j(\grave{q}_j) \geq z_i(q_i)$ then $x_i^{\phi}(q;C) = x_i^{\phi}(q_{-j},\grave{q}_j;C)$.

Lemma 3 (Lemma 2 in Moulin & Shenker [22]): *Let $h_1(\lambda), h_2(\lambda)$ be two increasing and strictly convex functions from R_+ onto itself that coincide up to λ_0:*

$$h_1(\lambda) = h_2(\lambda) \qquad \text{for all } \lambda, \quad 0 \leq \lambda \leq \lambda_0$$

Then for every utility function u_i in U_i, the (unique) maximizers of $u_i(h_k(\lambda),\lambda)$ on R_+, denoted by λ_k, $k = 1,2$ are on the same side of λ_0:

$$\lambda_1 \geq \lambda_0 \iff \lambda_2 \geq \lambda_0, \quad \lambda_1 = \lambda_0 \iff \lambda_2 = \lambda_0.$$

Proof of theorem:

Fix a cost function C satisfying the assumptions of theorem 1. Let the domain of utility functions representing the preferences satisfying the assumptions be U. Let the the set of alternatives be $A \equiv \{(q,x) : q \in [0, q^{\max}]^N, x \in R_+^N$ and $\sum_{i \in N} x_i = C(q)\}$.Consider a fixed path ϕ and the associated fixed path social choice function $\xi^{x^\phi} : U \to A$ which allocates the outcome corresponding to the unique NE of $\Gamma(x^\phi, u)$ to the preference profile u. Consider $D_i = \{0\} \cup \{t \in R_+ | \phi_i'(t)_-$ is positive$\}$[21] and $z_i(q_i) = \min[t | \phi_i(t) \geq q_i]$ (see figure 3 above for such an example of z_i). We claim that a function $a : \times_{i \in N} D_i \to A$ which is defined as follows is a generalized serial function and the associated generalized serial SCF $\zeta^a = \xi^{x^\phi}$. Let $a_i(z) = (q_i(z), x_i^\phi(q(z)))$ for all i, where $q_i(z) = \phi_i(z_i)$ and $q(z) = (\phi_1(z_1), \phi_2(z_2), ..., \phi_n(z_n))$. We will now prove the following three properties of a using lemma 2 and lemma 3 above and the assumption on preferences.

$\forall z \in \times_{i \in N} D_i$ and $\forall \lambda \in \mathbf{L}$:

(1)$z_i \leq z_j \Rightarrow a_i(z) = a_i(z_{-j}, s_j), \forall s_j \in [z_i, \infty]$,

(2)$\lambda.a_i(z_{-i}, s_i)$ has a unique maximizer $\overline{s_i}$, $\forall i$

(3)If $\overline{s_i}$ is the unique maximizer of $\lambda.a_i(z_{-i}, s_i)$ then $\overline{s_i}$ is also the unique maximizer of $\lambda.a_i(z'_{-i}, s_i)$ $\forall z'$ st., $\forall j \neq i$, $MIN[z'_j, z_j] < \overline{s_i} \Rightarrow z'_j = z_j$

First thing to notice is that even though the domain of "a" is not the same as in the original definition, the properties of "a" is retained exactly. This is so because for all i, the D_i is order-isomorphic to R_+ given D_i is concatenation of open-closed intervals with "0" included. Now we will show the above three properties one by one. To see that property 1 is true, notice that z_i uniquely defines $q_i(z) = \phi_i(z_i)$ which is independent of z_{-i}. Also, part (c) of the lemma 1 implies that $x_i^\phi(q(z)) = x_i^\phi(q(\grave{z}))$ $\forall z'$ st., $\forall j \neq i$, $MIN[z'_j, z_j] < \overline{s_i} \Rightarrow z'_j = z_j$. Property 2 is a consequence of part (a) of lemma 1 and the linearity of preferences. We first notice that strict convexity of $x_i^\phi(q; C)$ in q_i and linearity of preferences which are increasing in q_i and decreasing in x_i ensures a unique maximizer q_i^*. But then, there will a unique z_i^* for this q_i^* by the definition of z_i. Property 3 is a bit more subtle and the proof is as follows. Consider two points z and z' in $\times_{i \in N} D_i$. Consider a coordinate i. Let, $\forall j \neq i$, $MIN[z'_j, z_j] < \overline{s_i} \Rightarrow z'_j = z_j$. Consider $\lambda \in \{R_+ \times R_-\}/\{0\}$. Let $\bar{s}_i = \underset{s_i \in D_i}{\arg\max} \lambda.a_i(z_{-i}, s_i)$ and $\tilde{s}_i = \underset{s_i \in D_i}{\arg\max} \lambda.a_i(z'_{-i}, s_i)$. Let's call $\{a_j(z_{-i}, \bar{s}_i)\}_{j \in N} = \{(\bar{q}_j, \bar{x}_j)\}_{j \in N}$ and $\{a_j(z'_{-i}, \tilde{s}_i)\}_{j \in N} = \{(\tilde{q}_j, \tilde{x}_j)\}_{j \in N}$. From part (c) of lemma 2 we know $x_i^\phi(\bar{q}_{-i}, q_i; C)$ & $x_i^\phi(\tilde{q}_{-i}, q_i; C)$ coincide for all $q_i \in [0, \bar{q}_i]$. Also, we know from part (b) of lemma 2 that $x_i^\phi(\bar{q}_{-i}, \cdot; C)$ & $x_i^\phi(\tilde{q}_{-i}, \cdot; C)$ both are strictly convex in q_i. By the definition of $\bar{s}_i$ it follows that $\bar{q}_i = \underset{q_i \in [0, q^{\max}]}{\arg\max} \lambda \cdot (q_i\ ,\ x_i^\phi(\bar{q}_{-i}, q_i; C))$. But then from lemma 3 we must have $\bar{q}_i = \tilde{q}_i$. Finally to conclude the proof we notice, q_i being one to one function of z_i implies that $\bar{s}_i = \tilde{s}_i$. Also the way we have defined ζ^a and ξ^{x^ϕ}, they coincide.

■

[21] $\phi_i'(t)_-$ is the left hand derivative of ϕ_i at t.

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